STUDY AND SIMULATION OF THREE DIMENSIONAL FLOWS AT HIGH SUPersonic speeds FOR UTIAS 3-D EULER CODE VALIDATION WITH SPECIFIC EMPHASIS ON COMPRESSION LIFT APPLICATIONS FOR SUPERSONIC TRANSPORT AIRCRAFT.

ESC499 Thesis

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Abstract

The development and design of a successful supersonic transport (SST) aircraft faces many technical challenges. This thesis addresses the aerodynamic challenge of supersonic efficiency in an attempt to increase the L/D ratios of SST aircraft designs and thereby their ranges. The current focus for achieving this goal is the concept of compression lift which has never been thoroughly applied to the design of a SST. The theory behind compression lift is reviewed and compared to other similar aerodynamic theories and techniques applicable to increasing the L/D ratios of SST aircraft. The primary analysis tool, the UTIAS Euler 3-D code is validated by simulating a number of simple supersonic flow geometries and comparing them to the analytic results. The code has been found to give the correct qualitative results for all the simple geometries to within an acceptable quantitative error given the grid mesh sizes. Furthermore the effects of grid refinement were investigated for each flow scenario and it was found that the solutions converge in the limit of grid refinement, further qualifying the current simulation results. A preliminary analysis of a full SST aircraft configuration utilizing compression lift was then conducted using the UTIAS Euler 3-D code with promising results. The thesis concludes with a discussion of possible future research on the simulation of compression lift aircraft.
Acknowledgements

This thesis became possible due to the guidance, wisdom, and encouragement of my supervisors Professor Gottlieb and Professor Groth. Thank you to Professor Gottlieb for instilling a passion for supersonics that allowed me to pursue the current research with vigor and enthusiasm. Thank you to professor Groth for showing me that scientific computing and CFD are useful and interesting tools and for allowing me access to the UTIAS simulation code and resources. Finally I would like to acknowledge my family and friends who have supported me throughout my academic endeavors.

Graham Feltham
March 15, 2011
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<td>1-D</td>
<td>One Dimensional</td>
</tr>
<tr>
<td>2-D</td>
<td>Two Dimensional</td>
</tr>
<tr>
<td>3-D</td>
<td>Three Dimensional</td>
</tr>
<tr>
<td>AIC</td>
<td>Aerodynamic Interference Concepts</td>
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<tr>
<td>AMR</td>
<td>Automatic Mesh Refinement</td>
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<tr>
<td>CFD</td>
<td>Computational Fluid Dynamics</td>
</tr>
<tr>
<td>HPACF</td>
<td>High Powered Aerospace Computing Facility</td>
</tr>
<tr>
<td>HSCT</td>
<td>High Speed Civil Transport</td>
</tr>
<tr>
<td>HSR</td>
<td>High Speed Research</td>
</tr>
<tr>
<td>NACA</td>
<td>National Advisory Committee for Aeronautics</td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Administration</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary Differential Equation</td>
</tr>
<tr>
<td>SST</td>
<td>Supersonic Transport</td>
</tr>
<tr>
<td>SCAT</td>
<td>Supersonic Commercial Air Transport</td>
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<td>UTIAS</td>
<td>University of Toronto Institute for Aerospace Studies</td>
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<tr>
<td>cm</td>
<td>centimeters</td>
</tr>
<tr>
<td>m</td>
<td>meters</td>
</tr>
<tr>
<td>$V_\infty$</td>
<td>freestream velocity</td>
</tr>
<tr>
<td>$a_\infty$</td>
<td>freestream speed of sound</td>
</tr>
<tr>
<td>$M_\infty$</td>
<td>freestream Mach number</td>
</tr>
<tr>
<td>$L/D$</td>
<td>lift to drag ratio</td>
</tr>
<tr>
<td>$L/D_{\text{max}}$</td>
<td>maximum lift to drag ratio</td>
</tr>
<tr>
<td>$P$</td>
<td>pressure</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
</tr>
<tr>
<td>$T$</td>
<td>temperature</td>
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<tr>
<td>$M$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$s$</td>
<td>entropy</td>
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<tr>
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<td>internal energy</td>
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<tr>
<td>$\nu$</td>
<td>specific volume</td>
</tr>
<tr>
<td>$R$</td>
<td>specific gas constant</td>
</tr>
<tr>
<td>$c_p$</td>
<td>specific heat capacity</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>heat capacity ratio</td>
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<td>$i, j$</td>
<td>cell indices</td>
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<td>$u$</td>
<td>flow velocity in x-direction</td>
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<td>flow velocity in y-direction</td>
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<tr>
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<td>$M^*$</td>
<td>adiabatic characteristic Mach number</td>
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<tr>
<td>$l/d$</td>
<td>fineness ratio, length/diameter</td>
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<tr>
<td>$K$</td>
<td>similarity parameter $\frac{M}{l/d}$</td>
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<tr>
<td>$l$</td>
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<td>$R_{\text{max}}$</td>
<td>maximum body radius</td>
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<td>wingtip length</td>
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<td>angle</td>
</tr>
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<td>$\theta_s$</td>
<td>shock wave angle</td>
</tr>
<tr>
<td>$\theta_c$</td>
<td>semi-vertex angle</td>
</tr>
<tr>
<td>$\theta_{\text{sonic}}$</td>
<td>sonic line angle</td>
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<tr>
<td>$\beta$</td>
<td>shock wave angle</td>
</tr>
<tr>
<td>$C_{D_f}$</td>
<td>coefficient of skin friction</td>
</tr>
<tr>
<td>$\rho_\infty$</td>
<td>atmospheric density</td>
</tr>
<tr>
<td>$C_L$</td>
<td>coefficient of lift</td>
</tr>
<tr>
<td>$C_D$</td>
<td>coefficient of drag</td>
</tr>
<tr>
<td>$R_{a}$</td>
<td>aircraft range (m)</td>
</tr>
<tr>
<td>$\text{Re}$</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>$c_t$</td>
<td>thrust-specific fuel consumption</td>
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<tr>
<td>$S$</td>
<td>aircraft wing area</td>
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<td>final aircraft weight</td>
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<tr>
<td>$W_0$</td>
<td>initial aircraft weight</td>
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<td>angle of attack</td>
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<td>$W$</td>
<td>wing</td>
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<tr>
<td>$B$</td>
<td>body</td>
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<tr>
<td>$WB$</td>
<td>wingbody</td>
</tr>
<tr>
<td>$M_n$</td>
<td>normal component of flow Mach</td>
</tr>
<tr>
<td>$\theta_{\text{max}}$</td>
<td>max body half-angle for attached shock</td>
</tr>
<tr>
<td>$V_y$</td>
<td>component of velocity in y-direction</td>
</tr>
<tr>
<td>$a^*$</td>
<td>adiabatic characteristic sound speed</td>
</tr>
<tr>
<td>$C_p$</td>
<td>pressure coefficient</td>
</tr>
<tr>
<td>$q$</td>
<td>dynamic pressure</td>
</tr>
<tr>
<td>$d$</td>
<td>body diameter</td>
</tr>
<tr>
<td>$\theta_{\text{sweep}}$</td>
<td>wing sweep angle</td>
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1 Introduction

In the commercial airline industry passengers place speed along with safety, cost, reliability, and comfort as top priorities. Airline transportation speed has a significant impact on personal mobility, business interactions, and passenger attitude and its improvement has been a major design goal since the first days of commercial air transportation. With the introduction of faster swept-wing jet aircraft in the late 1950s the aviation industry along with several governments began to develop the next generation of commercial aircraft, a supersonic transport (SST) [1]. According to Chambers [1] ‘no other undertaking in the aeronautics research activities of the NASA, or its predecessor, the NACA, approaches the magnitude of human and monetary resources expended on the conception, development, and assessment of supersonic civil airplane configurations’. The chronology of SST research including programs, NASA research, and major milestones is shown in figure 60 in appendix A.

As will be shown in section 2 a large number of SST aircraft have been designed but only a fraction have made it off the drawing board and even then only entered service for a brief period. In order to better understand the reasons behind this a brief review of the most important technical challenges facing a successful SST design is presented in the next section. In addition the methods and tools of SST analysis and design are reviewed in the next section to better understand the past, present, and future of solution techniques.

The purpose of this thesis is to address one of the SST technical challenges using a pioneering concept developed by NACA researchers in the 1950s combined with a newly developed simulation code that may potentially improve the next generation of SST aircraft. The application of this concept to the design of SST aircraft has not been thoroughly applied and investigated even though it has shown great potential in improving supersonic aircraft performance. In addition to investigating the application of this concept this thesis has tested, validated, and applied the UTIAS 3-D Euler code to the design of supersonic aircraft, developing a useful tool for future SST research.

1.1 Technical Challenges in SST Development and Design

The largest technical challenges to the development of an SST involve the ability to meet all the mission requirements in an efficient, economically viable, and environmentally acceptable manner [1]. Along with an array of technical challenges SST aircraft also face an impressive list of nontechnical issues [2], only a few of the technical issues will be addressed however it is important to note that the successful design and operation of a SST is far from a solely technical endeavor.

Aerodynamic Challenges  The primary concern is efficient aerodynamic performance at supersonic speeds. With the introduction of wave-drag at supersonic speeds the dominant aerodynamic challenge is drag reduction. The geometric shape of the airframe along with the integration of the propulsion system must provide a sufficient lift/drag (L/D) ratio during supersonic and subsonic cruise as well as during takeoff and landing. Current subsonic jet transports can exhibit L/D ratios of 19 at cruise conditions while the most efficient SST designs have only achieved L/D ratios of less than 10. However highly swept configurations exhibit decreased aerodynamic efficiencies at off-design conditions such as subsonic...
cruise, loiter, and takeoff and landing. The Concorde at subsonic loiter and approach has a L/D ratio of \( \approx 4 \). Because the flight envelope of a SST is much larger than conventional subsonic transports it is difficult to satisfy the efficiency requirements of each flight regime with a single design [1].

**Stability and Control Challenges** The use of highly swept wings can result in compromises and trade-offs in the configuration design that can significantly decrease supersonic aerodynamic performance. The low speed longitudinal stability and control characteristics of highly swept arrow wings are usually unacceptable due to flow separation and may require modification to a less efficient supersonic shape (delta wing). For adequate pilot visibility during landing a ‘drooped-nose’ configuration may be employed with subsequent aerodynamic and weight penalties. The slender shape of SST aircraft creates flying quality issues and requires unconventional control augmentation systems for roll response and lateral-directional handling [1].

**Payload Challenges** To achieve adequate payload capability the airframe and propulsion individually and when integrated must minimize weight. In addition due to a large portion of off-design flight the aircraft must carry a large fuel reserve to meet contingencies during normal operations. Current subsonic transports typically have a 25-percent payload fraction while a representative SST has only a 6-percent payload fraction [1].

**Material Challenges** The selection of aircraft material is highly dependant on aerodynamic heating [3]. Depending on the selection of the design cruise Mach number the aircraft may have to incorporate non-conventional structures and materials. Conventional aluminum materials are not able to withstand the temperatures at cruise speeds above Mach 2.5 possibly resulting in the use of heavier materials such as steel or more exotic and expensive alternatives [1].

**Propulsion Challenges** The engines of a SST must operate efficiently over the subsonic and supersonic flight envelopes. Efficient engine cycles for supersonic cruise usually lead to low bypass ratio engine configurations which create extremely high noise levels if heavy noise suppressors are not employed. Takeoff noise levels must comply with noise regulations and has proven difficult for the low L/D, highly swept, long takeoff run SST aircraft [1].

**Sonic Boom Challenges** The shock waves produced by supersonic SST aircraft can lead to damage and civilian complaint over populated areas. Currently civil supersonic flight over land is prohibited by law and severely impacts the economic feasibility and operational flexibility of SST aircraft [1].

### 1.2 Methods and Tools for SST Development and Design

A primary result of the current research is the development and validation of a new SST research tool, the UTIAS 3-D Euler code. To place this new tool in the context of previous supersonic tools relevant to SST design, the following review has been performed.
Analytical Tools and Theory  A large body of analytical theory and solution techniques for a wide range of supersonic flow scenarios has been developed. An excellent presentation of the fundamentals are given by a number of introductory aerodynamics texts [4, 5, 6]. With respect to the application of SST design a number of relevant analytical theories have been developed for simplified analysis. A number of these theories have been reviewed and are the topic of discussion in section 3, in particular the NACA theory of compression lift was the inspiration for the current research effort.

Wind Tunnel Testing  To experimentally observe the supersonic characteristics of aircraft including aerodynamic coefficients, flow field characteristics, and shock/expansion wave patterns the aeronautical engineer has two choices: (1) conduct flight tests using the actual aircraft or (2) run wind tunnel tests on a scale-model of the aircraft. Because flight tests are costly and may be dangerous if non-conventional aircraft configurations are investigated wind tunnel tests have become standard practice. Since their introduction in the mid-1930s by Adolf Busemann and Prandtl most modern aerodynamic laboratories have one or more supersonic wind tunnels [4]. Wind tunnels measurements are often used as comparison to analytical models and aerodynamic theory, examples of this will be presented during the discussion of the relevant SST theory, analysis, and research in section 3.

Method of Characteristics  A classical method of solving the governing equations in a completely supersonic flow situation which was developed in 1929 [4]. Most generally used for two-dimensional, irrotational, steady flow but can be used for three-dimensional, rotational, and unsteady flows as well. A classical example of its implementation is in the design of supersonic nozzles. The method of characteristics is an older numerical development and has been generally replaced by CFD [5].

Slender Body Theory  Also known as linearized supersonic flow and assumes only small perturbations throughout the flowfield [4]. This is an approximate analytical method which is similar in nature to the method of characteristics and is useful for simple geometries that are approximately shock-free and irrotational [6]. This method can be applied when the freestream Mach angle is greater than the maximum angle of flow deflection [7]. As an example of its application the Concorde aircraft was designed using slender body theory [2].

Panel Methods  Panel methods are useful tools for simulating full three-dimensional arbitrary configurations. Generally higher-order panel methods are used which are based on the solution of the linearized potential flow boundary-value problem at subsonic and supersonic Mach numbers. The results are generally not valid for cases where viscous effects and separation are dominant [8]. Panel methods have been used extensively by industry, for example the Boeing code PANAIR (A502), which has now been replaced by CFD methods and henceforth become commercially available [9].

Wire-Frame Techniques  High speed computers were first added to the design of SST aircraft by Harris in his adaptation of the Boeing code for prediction and analysis of zero-lift supersonic wave drag. Harris’ work was the first ‘wire-frame’ rendering of aircraft configurations for analysis [1]. Wire-frame
techniques are used to parameterize the geometry of the aircraft and are useful for geometry investigation and optimization when coupled with any aerodynamic numerical technique [10] an example wire-frame representation of the SCAT-15F SST is given in figure 61 in appendix A.1.

**CFD** Although some of the previous techniques may be considered to fall under the heading of ’Computational Fluid Dynamics’ such as the method of characteristics, most authors consider that CFD is represented by finite difference and finite volume techniques [4]. These methods numerically solve the exact fundamental nonlinear governing equations of continuity, momentum, and energy which allows the engineer to obtain solutions to fluid dynamic problems that have no closed form solutions. When solving mixed supersonic/subsonic flows a time dependant technique is often incorporated however for steady supersonic flow situations a non time-dependant technique is sufficient. The difference lies in the fact that steady supersonic flowfields are governed by hyperbolic differential equations whereas steady subsonic flowfields are governed by elliptical differential equations [5]. CFD methods are becoming very powerful and popular and are often required to solve complicated flow situations for which some examples relevant to SST research are presented below.

- The use of CFD has become more important because wind-tunnel models have become larger to incorporate realistic features such as twist, chamber and nacelles. Larger models require the measurement of the shock signatures at closer and closer distances [1].

- Because the traditionally used sonic boom theory is only valid at mid to far-field distances, CFD methods are the only means of generating a near-field signature, one that can be compared directly with wind-tunnel data [1].

- The HSCT-H was the first SST to use powerful non-linear computational fluid dynamic (CFD) methods in the analysis and design stages [1].

The current research focuses on steady, inviscid, supersonic flows for which time-dependant and non-time dependent CFD techniques developed by researchers at UTIAS are used to solve the governing Euler equations for a polytropic gas.

### 1.3 Objectives of Current Research

As shown in the previous discussion the development of an lasting economically feasible, technically viable SST involves a large number of technical challenges. The current research is focused on the aerodynamic challenges and in particular the analysis and possible improvement of the supersonic aerodynamic efficiency (L/D ratio) of SST aircraft designs. According to Obe [11] the highest L/D ratio of an investigated SST design in the Mach range 1.5-3.5 is just above 10, subsequent designs should aim to surpass this as an aerodynamic goal. This is of primary importance because the range of the aircraft is dictated by the famous Breguet equation [12] which is proportional to the aircrafts L/D ratio$^1$,

$^1$represented by the equivalent lift and drag coefficients
\[ Ra = 2 \sqrt{\frac{2}{\rho_\infty S} \frac{1}{c_t} \frac{C_{D}^{1/2}}{C_{D}} (W_0^{1/2} - W_1^{1/2})} \]  

In 1985 a committee of government, industry, and academic experts reviewed the study of SST aircraft and specified goals for future research, one of which was ‘To Attain Long-Distance Efficiency’ or simply put ‘To Increase the Range of SST Aircraft’ [13]. Therefore the challenge of increasing the range of SST aircraft is an important focus in SST development. In order to achieve the final goal of SST L/D ratio calculation and improvement the research was decomposed into a number of steps which are roughly equivalent to the section division of this report. Due to time limitations a number of the presented objectives have not yet been completed however they have been included to provide a full outline of the research methodology and perhaps to guide future research on the topic.

**Primary Research Objectives**

1. Investigation of past/current SST configurations and primary concerns in development.
   - research into problems facing current and future SST aircraft with emphasis on required aerodynamic efficiency.
   - review of SST aircraft designs with emphasis on airframe geometry and engine placement.

2. Investigation of compression lift phenomenon and discussion of supersonic flows.
   - discussion of potential supersonic and hypersonic applications and flow characteristics.
   - discussion, replication, and comparison of original NACA results on compression lift.
   - discussion of interference concepts applicable to supersonic aircraft design.
   - discussion of arrow wing designs and results.
   - discussion of Newtonian impact theory and results.
   - presentation and discussion of aircraft geometries to be investigated.

3. Simulation of supersonic flow past simple geometry using Euler 3-D code
   - discussion of UTIAS Euler 3-D code, including AMR and grid refinement.
   - discussion of shock wave theory and necessary thermodynamics.
   - discussion of supersonic plate flow with relevance, theory, and equations.
   - comparison of Rankine-Hugonoit analytical solution for plate to Euler 3-D simulation.
   - discussion of supersonic wedge flow with relevance, theory, and equations.
   - comparison of Meyer analytical solution for wedge to Euler 3-D simulation.
   - discussion of supersonic cone flow with relevance, theory, and equations.
   - comparison of Taylor-Maccoll analytical solution for cone to Euler 3-D simulation.
- discussion of supersonic axisymmetric flow with relevance, theory, and equations.
- comparison of Newton and slender body theory of Shapiro analytical solutions for axisymmetric to Euler 3-D simulation.
- extension of plate flow to blunt bodies with relevance, theory, and equations.
- comparison of approximate cone flow and tangent sphere-cone flow for blunt bodies to Euler 3-D simulation.

4. Simulation of supersonic flow past SST aircraft configurations.
- simulate supersonic flow past flat-top and flat-bottom aircraft configurations.
- show the effects of an aircraft fuselage when added to the main wing.
- investigate compression lift effects on the XB-70 aircraft configuration.
- determine and compare L/D ratios for the XB-70 aircraft and a similar size arrow-wing SST.
- investigate the effects of drooping wingtips and wing anhedral on L/D ratios.

Possible Extensions of Current Research Because the subsonic performance of an SST must be at least comparable to supersonic cruise for overland trips a possible extension of the current research is to investigate and possibly improve the subsonic L/D ratios for SST aircraft designs without sacrificing the supersonic L/D ratios [2].

2 Review of Past SST Aircraft Designs

In order to gain some insight into the aerodynamic design of SST aircraft and its progression over the years the most important SST configurations are reviewed with emphasis on the airframe designs, operating Mach numbers, propulsion integration schemes, and development methods and tools.

XB-70 Valkyrie In the late 1950s the XB-70 supersonic bomber was a joint effort between the United States Military and NASA and became the basis for a civil transport program. The aircraft operated at a cruise Mach of 3.0 and was designed before high-speed computers or automated procedures were involved in the design process [1]. The XB-70 incorporated many advanced aerodynamic features that helped provide the capability of meeting the range requirement. In particular the positioning of the wing and fuselage as shown in figure 1 resulted in compression lift with folding wingtips further increasing this effect [14]. The compression lift concept will become an integral part of the current research and is discussed more thoroughly in section 3.

SCAT Configurations During the 1960s a significant undertaking into the design of a Supersonic Commercial Air Transport (SCAT) configuration was undertaken by NASA. Over 40 concepts were developed but only the four shown in figure 2 were chosen for more detailed analysis, these being the SCAT-4, SCAT-15, SCAT-16 and SCAT-17. SCAT-4 was an elegant arrow wing design, SCAT-15 included an
innovative variable-sweep arrow wing design, SCAT-16 was a more conventional variable-sweep design, and SCAT-17 was a canard configuration with a fixed delta wing. SCAT-16 and SCAT-17 were chosen as the most promising final designs. However a modification of the SCAT-15, the SCAT-15F was further investigated by Boeing alongside its 2707 series designs. The SCAT-15F showed unprecedented aerodynamic efficiency at supersonic speeds with a L/D ratio of 9.3 but suffered from poor stability and control characteristics and poor subsonic aerodynamic performance. The SCAT-15F along with the Boeing 733, 2707 and Lockheed L-2000 designs were among the first to apply high-speed digital computers to the aerodynamic design of the airframes [1].

**Figure 1:** XB-70 Supersonic Bomber.

Boeing Models 733 and 2707  At the same time as the development of the SCAT configurations Boeing was developing and refining its designs for a commercial SST. The initial Model 733 series was a variable-sweep design as shown by figure 3 but quickly transitioned into a delta wing and then a double-delta wing in the Model 2707 series. The 2707-300 was selected as the final design however it was canceled due to insufficient range capability [2].

**Lockheed L-2000**  In competition with Boeing for a supersonic transport contract Lockheed developed the Model L-2000. The L-2000 was a double-delta configuration that had an improved L/D ratio of 8, much higher than the initial Boeing designs, see figure 62 in appendix A. However the revised Boeing 2707-100 beat out the L-2000 because the design boasted a L/D ratio of 8.2 and had a larger passenger capacity.
The Tu-144 was designed for service in the Russian airline industry and became the world's first supersonic transport in 1968. The Tu-144 was a sleek double-delta aircraft designed to cruise at Mach 2.0, see figure 63 in appendix A. In order to maintain pilot visibility during takeoff and landing the Tu-144 has a drooping nose design [11]. The Tu-144 entered service for only 3 years before retiring and becoming a testbed aircraft for NASA supersonic research [1].

The Concorde is the only SST to operate successfully in commercial service. The Concorde was an all-aluminum aircraft that incorporated a thin ogee-delta wing and was designed to cruise at Mach 2.2, see figure 64 in appendix A. The Concorde was designed using linearized potential flow theory (slender body theory) [2]. Similar to the Tu-144 the Concorde uses a drooping nose to maintain pilot visibility [1]. The Concorde is currently retired after a service life of 27 years with a history of only one crash in July 2000 [18].

The HSCT reference H was the baseline aircraft under development during the High Speed Research (HSR) program at NASA in the 1990s. The HSCT-H was the culmination of research and design during the HSR from which a number of predesign designs were improved, see figure 66 in appendix A. The HSCT-H was a sleek double-delta with four aft-mounted pod engines and was designed to cruise at Mach 2.4, see figure 65 in appendix A. The HSCT-H was the first SST to use powerful non-linear computational fluid dynamic (CFD) methods in the analysis and design stages [1].

From the reviewed SST designs and the analysis of the progression in the aerodynamic design of SST aircraft a number of conclusions may be drawn. The arrow wing design exhibits high supersonic efficiencies but suffers aerodynamically and has poor stability and control characteristics at low speeds. The delta wing design is very popular and solves the stability and control issues of the arrow wing with sacrificed supersonic performance. A popular variant of the delta wing design is the double-delta and ogive-delta designs. Variable sweep wings have been investigated but suffer from decreased performance resulting in all final aircraft designs with fixed wings. The fuselage is kept slender and integrated into the main wing.
The engines are always aft-mounted and all recent SST aircraft involve individual pod mounted designs. It is interesting to note that in newer designs while the majority of the airframe is highly integrated resulting in a ‘sleek’ design the engines are distinctly separated, this is a major design break from the older XB-70 configuration.

To gain insight into the justification behind the aerodynamic design choices and progression of SST designs, the next section reviews a number of aerodynamic theories and investigations directly applicable to the design of SST aircraft. In particular the distinct integrated fuselage/propulsion delta wing design of the XB-70 aircraft will be investigated and compared to the more recent arrow-wing or double delta ‘sleek’ aircraft configuration with separated individual engine nacelles.

3 Review of SST Theory, Analysis, and Research

As previously discussed in the introduction (section 1) the current research was inspired by the aerodynamic theory of compression lift that was first applied to the design of supersonic aircraft by NACA researchers A. J. Eggers and Clarence A. Syvertson [3]. This theory was the basis for the design of the XB-70 supersonic bomber presented in section 2 and was predicting a significant increase in the supersonic L/D ratios. To better understand the phenomenon of compression lift and its applications the original NACA research was reviewed and reproduced. Furthermore a number of additional theories developed to increase the supersonic L/D ratios of aircraft have been reviewed to provide comparison to the theory of compression lift. Even though compression lift is the focus of the current research it is important to understand the other competing theories and possibilities to better judge the applicability and potential of compression lift applications. Furthermore a number of additional considerations directly applicable to the current research became apparent after considering the other competing theories.

3.1 NACA Original Research of Compression Lift

The challenge of designing an aircraft developing high L/D ratios at supersonic speeds using compression lift was addressed by Eggers et al. [3, 19] using the developed fundamental ‘momentum principle’ which states that the components of the aircraft should be individually and collectively arranged to impart maximum downward momentum and minimum forward momentum\(^2\). In addition, simplified considerations of aerodynamic heating, structural weight, and stability and control were considered in the design of the aircraft configurations, a summary of the restrictions due to these considerations is given.

Aircraft Design Restrictions

- The aircraft wings are highly swept with blunt leading edges.
- The aircraft body has a blunt nose.
- The aircraft body is a major lifting element.

\(^2\text{this may also be considered as favorable interference effects}\)
- The aircraft body is shaped to stabilize the vehicle in flight.
- All other stabilizing surfaces are located on the windward side of the aircraft.
- The aircraft is of a slender design.
- The aircraft must minimize pressure drag.

The effects of these restrictions result in several design decisions regarding the general aircraft configuration. Because the aircraft is slender the maximum L/D ratio will occur at low $\alpha$ and therefore a thin wing was selected. To minimize pressure drag and stabilize the aircraft continuously enlarging bodies of revolution were chosen for the aircraft body, specifically a conical body shape was selected. These bodies are known to have low drag at high supersonic speeds and their flare effect contributes to stability [3]. This resulted in the general configuration of a conical body aircraft with thin swept wings located midbody of the aircraft as shown in the left of figure 4.

The 'momentum principle' was then invoked to make three further design modifications. Considering the left configuration in figure 4 at supersonic speeds at low $\alpha$ the upward momentum generated by the pressure forces on the top of the body cancel the downward momentum generated by the pressure forces on the bottom of the body. If the upper portion of the body is removed to create a flat-top configuration a net gain in upwards momentum is realized. The wing now serves the function of preserving the downward momentum of the disturbed air by the lower body as shown in the right of figure 4. Eggers suggests that configurations of this type can achieve L/D ratios from $15 - 20\%$ higher than those of corresponding flat-bottom or symmetrical type aircraft [19]. Because the flow is supersonic the body can impart downward momentum in the region between its surface and the shock wave generated by the nose. For this reason the wing should only extend as far as the shock wave, because any portion of the wing ahead of the shock will not contribute to net downward momentum but will contribute to net forward momentum through friction forces. The wing leading edge therefore follows along the shock angle from the body nose as shown in figure 5. The trailing edge connects the aft-most part of the body but its angle is not determined through elementary reasoning as is the case of the leading edge. Because the body imparts not only downward but also lateral momentum to the air which creates a cross-flow along the wings the wing-tips should be deflected downwards to deflect this momentum downwards as shown in figure 6. The result of the previous discussion is the final aircraft configuration shown in figure 7.

**Figure 4:** Application of momentum principle resulting in half-body design from [3].
Development of Analytic Theory and Equations  To calculate the theoretical L/D ratios for aircraft configurations as shown in Figure 7 a number of simplifying assumptions were used to simplify the equations. The wings were idealized as flat plates and the bodies are simplified as one-half a body of revolution. The lift is assumed to vary linearly with $\alpha$ and the drag due to lift varies as the product of lift and $\alpha$. The drag is equal to the pressure drag of the half-body, the friction drag of the whole aircraft, and the induced drag of the wings. The contributions from base drag are neglected because it is normally only a small percentage in unpowered flight and may be positive or negative in powered flight [3].

In order to appreciate the aerodynamic theory the full development of the equations from first principles was performed in the current research and followed alongside the development provided by A. J.
Eggers in [3]. A short program was written in Matlab® to integrate the pressure distributions along the wings due to the half-body for a range of semi-vertex angles which were taken from tables provided by Kopal [20].

**Results, Comparison and Discussion** A preliminary analysis of a flat-top configuration with a 5° semi-vertex angle half-cone body for a range of Mach numbers and parasitic drag values was performed using the developed equations. The wing leading edge is determined by the shock wave angle and the trailing edge is formed by a straight line swept back from the body base an intersecting the leading edge 1.4 body-lengths aft of the nose. It is important to note that the planform area changes with changing Mach number because the wing leading edge follows the changing shock wave angle.

The results of Eggers et al. [3] is compared to the current results in figure 8. Because both results followed from the same governing equations and derivation it is surprising to note the significant differences. These are most likely a result of the different calculation techniques with the results of Eggers et al. being calculated and interpolated by hand and the current results being calculated and plotted using Matlab®.

In both cases it was found that increasing Mach number and/or skin friction reduced the maximum L/D ratio of the aircraft. For a flat-top wing-body aircraft with a parasitic drag coefficient $C_{D_f}$ of 0.005 the maximum L/D ratio at Mach 3 is approximately 9-10 when considering the results of Eggers et al. and the current results as shown in figure 8. The L/D ratio for this aircraft decreases with increasing Mach number to 6 at a Mach number of 7. The results of the analysis show that it is advantageous to fly at lower Mach numbers and to decrease the parasitic drag $C_{D_f}$ as much as possible to obtain the highest L/D ratio.

In addition an investigation of the effects of the body semi-vertex angle on L/D ratios at Mach 5 was investigated by [3]. It was found that a L/D maximum occurred for semi-vertex angles in the range $4° - 8°$ for all parasitic drag values, see figure 67 in appendix B.1 for the full results.

Because of the approximate nature of the analysis and to check the aerodynamic theory, a wind tunnel investigation was performed on several flat-top wing-body combinations by [3]. The results for the aircraft shown in figure 9 are given in figure 68 in appendix B.1 and show that the maximum L/D at Mach 5.0 is 6.65 which is very close to the theoretical prediction of 6.85. The effects of deflecting the wingtips was also investigated and are shown in figure 68. It was found that deflecting the wingtips decreased the L/D ratios slightly. A more complete experimental investigation involving both flat-top, flat-bottom, and symmetric configurations was performed by [19] with the results given in figure 69 in appendix B.1 and clearly shows that the flat-top configurations develop the high L/D ratios over the entire investigated Mach number range. Furthermore the effects of drooping wingtips on flat-top configurations was further investigated by [19] and confirmed the prior results of [3] that deflecting the wingtips decreases the maximum L/D ratio of the aircraft at high supersonic speeds, see figure 70 in appendix B.1 for the complete results.

**Application to the XB-70 Aircraft** When the XB-70 aircraft was being designed the aerodynamic engineers became aware of the research and results of Eggers et al. [3] discussed previously and incorporated the use of compression lift into the design to achieve a sufficient L/D ratio [21, 14]. Compression lift was achieved by positioning the wing to use the pressure field behind the shock wave generated by the engine inlet portion of the fuselage body as shown in figure 10. This was possible by the supersonic inlet which must create a shock wave to slow and compress the air for subsonic combustion. The first shock
was generated by the fixed wedge of the duct inlet. The pressure rise behind this shock was superimposed underneath the wing such that a significant portion of the aircrafts weight was supported by the pressure rise from the shock wave. The XB-70 also included folding wingtips which provided directional stability at supersonic speeds and increased the gain from compression lift. The shock waves from the inlet wedge reflected off the wingtips and increased the pressure rise under a portion of the aircrafts wing. The increased compression lift effect allowed the XB-70 to cruise at the same angle of attack despite the reduced wing area and change in aspect ratio, taper ratio, and other wing properties [14].

Application to Current Research  The results from the analytical theory, experimental wind tunnel testing and design of XB-70 all look promising in increasing the L/D ratios of SST aircraft. Since all supersonic aircraft will create a shock wave by the nature of the flow situation and engine inlets often require significantly strong shock waves to operate properly it seems reasonable to attempt to harness the otherwise “wasted” energy of the shock wave into useful lift. The current research is therefore an attempt to apply the concept of compression lift as described to the design of a new generation SST aircraft with potential for increasing the supersonic L/D ratio. The first step in this direction is the successful simulation of the flat-top, half-body configurations as presented by Eggers et al. by the UTIAS Euler 3-D simulation

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Figure 8: Comparison between $L/D_{max}$ calculations from current research and NACA [3] for $5^\circ$ semi-vertex angle flat-top aircraft configuration, with planform changing with Mach number.
Figure 9: Experimentally investigated flat-top aircraft configuration from [3].

code which is the topic of section 5.

3.2 Application of Aerodynamic Interference Concepts (AIC) to Supersonic Aircraft

A review of possible methods to increase supersonic/hypersonic aircraft aerodynamic efficiencies was investigated by R. M. Kulfan [22] and included a review of the flat-top wing/body arrangements of Eggers et al. discussed previously. The review also included a large number of other possible designs including supersonic biplanes, ring wings, parasol wings, and wave rider concepts. The review focused on calculating the favorable aerodynamic interference effects associated with interactions of flow fields created by different aircraft components with the result being drag reductions or additional lift production. The methods used to calculate these effects were a combination of linear approximations and small-disturbance potential flow methods. The review concluded that a parasol wing design provided the most promising results but the results for the flat-top wing/body arrangements will be focused on for comparison to the previous discussion and results of Eggers et al. [3, 19].

Wave Producer Aircraft The flat-top wing/body aircraft arrangement is classified as belonging to the wave producer family. This type of aircraft is characterized by shock waves and expansion waves and the shock waves can be used to produce compression lift as mentioned previously. It is noted that wave producer aircraft are generally highly integrated, with volume, lift, and thrust producing components sometimes aerodynamically indistinguishable. It is also noted that the major drag contribution for this type of aircraft is wave drag, resulting from the large pressure changes due to the shock and expansion waves [22]. It is therefore of primary importance to determine if the application of compression lift through the use of well placed shock waves results in a greater lift increment than a wave drag increment to increase the L/D ratio, this is part of the primary investigation of section 5.

Results, Analysis and Discussion The analysis of Kulfan [22] uses a simple linear approximation for the increment in lift due to the superposition of the shock wave and the aircrafts wing. The results indicate that the increase in body wave drag is significant and that when the aircraft is at an angle of attack the wing slower surface pressures produce an unfavorable wing-on-body interference which diminishes the favorable lift increments.

University of Toronto
Figure 10: XB-70 compression lift design and effects from [14].

The analysis also included the effects of wing anhedral shown schematically in figure 11. It was found that the lift increment due to compression lift can be increased by wing anhedral shown in figure 12 but that the body wave drag also increased shown in figure 71 in appendix B.1.

Application to Current Research The results of Kulfan’s [22] analysis on flat-top wing/body aircraft give mixed messages with respect to the current research. The results indicate that the increase in wave drag outweighs the increment in favorable lift however the analysis of the lift increment was only a linear approximation and therefore may underestimate the true potential of compression lift. An important side-note is that the inclusion of wing anhedral may increase the effects of compression lift and may result in a further increase in the L/D ratios.

3.3 NASA Research on Arrow-Wing Design Supersonic Aircraft

As a comparison to the flat-top wing/body aircraft configuration the arrow-wing design was also briefly investigated. This aircraft type is important because it appears that most new SST designs are of the arrow-wing configuration ⁴. The arrow wing design is a more integrated design with the separate effects of the wing and fuselage being less distinct than in the flat-top wing/body design presented earlier. The arrow wing design has a significant fuselage nose ahead of the wing leading edge and generally incorporates

⁴see section 2 and the corresponding appendices for aircraft diagrams and layouts supporting this claim
both a long slender fuselage and a thin, pointed, highly swept wing. The L/D ratios for these aircraft were both analytically calculated and experimentally found by Jorgensen [23]. An arrow wing with both a circular and elliptical fuselage cross-section were investigated as shown in figure 13.

The results from Jorgensen’s analysis [23] and experimentation shown in figure 14 show that the use of an elliptical body can result in a noticeably higher maximum L/D ratio than that obtained using a circular body. The results also indicate that the wing alone achieves the highest maximum L/D ratio and that adding the body to the wing causes the L/D ratio to decrease, increasing the body diameter almost linearly decreases the maximum L/D ratio for the aircraft as shown in figure 72 in appendix B.2. It was also found that the maximum L/D ratio was significantly decreased for adding volume into the body instead of the wing as shown in figure 73 in appendix B.2 and therefore for the greatest aerodynamic efficiency volume should be designed into the wing instead of the body.

It is interesting to note that the L/D ratios for the arrow-wing configurations with a body as shown in

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5 the analytical approach is very similar to that presented in [3] discussed previously

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Figure 11: Wingbody wave producer aircraft with different wing arrangements from [22].

Figure 12: Increment in lift due to wing anhedral from [22], only poor reproduction quality available.
Figure 13: Planform views of arrow-wing aircraft models, all linear dimensions are in inches, from [23].

Application to Current Research  Jorgensen’s results [23] on the L/D ratios of arrow-wing aircraft configurations leads to several helpful hints towards techniques that may be used to increase the possible L/D ratios of compression lift aircraft. It was found that an elliptical body produces a higher L/D ratio than a circular body and the reshaping the lower body in a compression lift aircraft in an elliptical manner should be investigated. Furthermore it was found that the addition of the body decreased the maximum L/D ratio and that increasing the body diameter linearly decreased the L/D ratio, for these reasons minimizing the body diameter while still creating a sufficient shock wave for compression lift should be investigated in the context of the current research.

3.4 NASA Research on the use of Newton’s Law of Resistance for Supersonic Aircraft Design

To conclude the development of aerodynamic theories and investigations of supersonic aircraft L/D ratios the results of Resnikoff [24] are discussed to make clear the importance of considering interference effects in the design of supersonic aircraft.

Resnikoff’s analysis [24] uses Newton’s law of resistance to calculate the L/D ratios of simple ge-
4 Simulation of Supersonic Flow past Simple Geometries

When developing a new analysis tool it is important to validate it using test cases. The UTIAS Euler 3-D code was newly developed and required validation before being applied to the analysis of SST aircraft configurations. A range of test cases spanning all flow dimensions (1-D, 2-D, and 3-D) were developed and the Euler 3-D simulations were compared to the accepted analytical results. Before discussing these test cases a brief introduction to the UTIAS Euler 3-D code as well as shock wave theory from which the analytic results are derived are presented in the next sections.
4.1 Introduction to UTIAS Euler 3-D Simulation Code

The UTIAS Euler 3-D simulation code was developed in C++ and runs on the HPACF at the University of Toronto. The code is powerful and versatile, with simple methods for creating multi-blocks meshes for a range of geometries and more complex methods for including grid refinement techniques such as Automatic Mesh Refinement (AMR) [25]. The code solves the Euler equations in three-dimensional space using either a time-accurate or non time-accurate approach. Because the Euler equations are the inviscid form of the full Navier-Stokes equations the dissipative effects of friction, thermal conduction, and diffusion are neglected. This approximation is best approached when the Reynolds number approaches infinity and the viscous effects are limited to the boundary layer which are good assumptions for high Mach supersonic flow which is the flow case for the current analysis [5]. In a more general sense the code is applicable to an Euler polytropic process defined by $PV^n = constant$, where a polytropic process is internally reversible and $n = 0$ is an isobaric process, $n = 1$ is an isothermal process and $n = \gamma$ is a reversible adiabatic process (isentropic). The current research focuses on the last of these cases where air is the working medium and $\gamma = 1.4$.

A simulation geometry is defined using structured quadrilateral blocks which are shown in comparison to an unstructured approach in figure 16(a). Structured meshes are used instead of unstructured ones because the simulation geometry is fairly and structured meshes are much simpler to implement. The blocks which define the grid are broken up into a number of cells which form the mesh with the hierarchy shown in figure 16(b) with an example implementation shown in figure 16(d). When using AMR the code only allows 2-1 cell boundaries between blocks as shown in figure 16(c) because the flow of information would be seriously degraded with a ratio higher than this [26].

When performing the solution the code can use first order, second order, or a combination (limiter type) approach. The limiter approach uses second order where large discontinuities do not exist because it is more accurate, but for large discontinuities second order is unstable so the limiter reverts to first order in those regions. The limiter approach generally selected for the current research is the venkatakrishnan approach which automatically selects first or second order where appropriate. The code selects regions of discontinuities by measuring gradients in flow variables such as $\rho, P, T$ [26].

**Figure 15:** Results of Newton’s theory of resistance for L/D ratios of supersonic bodies from [24].
4.2 Introduction to Shock Wave Theory and Analysis

When a body moves through air or another medium at a uniform speed that is greater than the speed of sound a shock wave is formed and remains fixed relative to the body [27]. A shock wave forms ahead of any body in supersonic flight and remains fixed relative to the body if the flow is steady. The shock wave will be detached for blunt bodies but may be attached for pointed shapes [28]. A shock wave is a very thin region. Shock waves occur in supersonic flow as a solution to the problem of the propagation of disturbances. The disturbances of a body in a flowfield tries to propagate everywhere, including upstream by sound waves. If \( V_\infty > a_\infty \) then the sound waves tend to coalesce a short distance ahead of the body and this forms a thin shock wave. Ahead of shock wave the flow has no knowledge of the body disturbance, behind the shock the flow is subsonic and the streamlines compensate for the body obstruction [5]. Across the shock wave a very abrupt change in density, pressure, and velocity occurs, shock waves are discontinuities across which flow properties change suddenly. The nature of shock waves can be seen experimentally as sharp lines in Schlieren and shadowgraph experiments and the governing equations were first properly

\[^6\text{thickness typically on the order of a few molecular mean free paths, typically } 10^{-5} \text{ cm for air at standard conditions}\]
formulated by Rankine [29]. Rankine was the first to show that within the shock wave a non-adiabatic process must occur and Hugoniot showed that in the absence of viscosity and heat conduction conservation of energy implies conservation of entropy in smooth regions and a jump in entropy across a shock [29]. Because of the pioneering work of both Rankine and Hugoniot a modern generic term has come into use for all shock equations, namely Rankine-Hugoniot equations. Another important contribution came later from Lord Rayleigh who pointed out the direction of changes across shock waves using the second law of thermodynamics with contributions from G.I Taylor [5]. The governing equations relate the properties of the flowing medium in front and behind plane shock waves. These equations were subsequently applied by Meyer to the situation of an inclined plate or wedge, resulting in the oblique shock equations [27].

Flows across shock waves are generally adiabatic but are not reversible and hence non-isentropic. Shock waves are non-isentropic due to large gradients inside the shock. In regions of large gradients the viscous effects and thermal conduction become important. These are dissipative, irreversible phenomenon which generate entropy. For a calorically perfect gas the stagnation temperature $T_0$ remains constant\(^7\). The total pressure $P_0$ decreases across a shock wave. Since the static pressure always increases across a shock wave it can be visualized as a thermodynamic device which compresses the gas [5]. In general the properties behind a shock must be solved for numerically, however if we assume a calorically perfect gas we can sometimes derive analytical solutions [5].

In a large number of applications the flows ahead and behind the shock wave can be considered isentropic. Furthermore if the flow is considered calorically perfect\(^8\) then the following isentropic flow relations apply\(^9\)

\[
\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1}\right)^\gamma = \left(\frac{T_2}{T_1}\right)^\left(\frac{\gamma}{\gamma-1}\right)
\]

\[
\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2
\]

\[
\frac{P_0}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^\frac{\gamma}{\gamma-1}
\]

\[
\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^\frac{1}{\gamma-1}
\]

Shock wave compression is very effective but inefficient. Comparing the effects of isentropic and shock wave compressions on the Hugoniot curve shown in figure 74 that for a given decrease in the gas specific volume the shock wave creates a higher pressure increase than the isentropic compression. However due to the entropy increase of the shock wave the shock compression is less efficient because the total pressure $P_0$ decreases [5].

A discussion on the supersonic flow past simple geometric bodies will be presented in the next sections. The most simple being presented first, starting with 1-D flow, and then progressing to 2-D and 3-D flow situations. In all cases we are considering stationary shock wave phenomena with the flowing medium being air and $\gamma = 1.4$.

\(^7\)this is a special case of the more general result that the total enthalpy $h_0$ is constant

\(^8\)constant specific heats, good approx for atmospheric air below 1000K which generally corresponds to $M_\infty < 5$ [5]

\(^9\)equation 3 only requires the more general case of adiabatic conditions
4.3 Plate Flow

Plate flow describes a 1-D flow situation with normal shock waves. Plate flow can be used to approximate detached normal shock waves on bodies with flat surfaces perpendicular to supersonic flow. This is important for supersonic aircraft because of the possibility of a large number of these regions over the aircrafts surface. Indeed normal shocks occur frequently in many supersonic flowfields [5]. Normal shocks are perpendicular to the flow with the flowfield being inherently one-dimensional. The flow $M_2$ behind the normal shock is always subsonic while $P_2, T_2, \rho_2$ are increased. Experimental images of supersonic flow past plate-like bodies in practical applications is given in figures 17(a), 17(b).

![Figure 17](image1.png)

(a) Cylinder at $M_\infty = 3.6$ in air [30].  
(b) Cylinder at $M_\infty = 2.77$ in CO$_2$ [30].

**Figure 17:** Shadowgraph and finite-fringe interferogram of supersonic flow past cylinders.

The flowfield situations in figures 17(a), 17(b) approximately correspond to the situation of a flat plate in supersonic flow. A well defined detached bow shock occurs ahead of the body for all values of $M_\infty$ with the separation distance decreasing with increasing $M_\infty$. If the area of the flat plate cross-section is sufficiently large and we consider the local portion of the shock wave corresponding to the center of the flat plate we have an approximate for a normal shock wave. This procedure and approximation is shown in figure 18 for a blunt body in which the flat plate is the limit\(^{10}\).

Normal shock waves occur frequently as part of many supersonic flowfields and is perpendicular to the flow by definition. The flow is supersonic ahead of the shockwave and subsonic behind it. The static pressure, temperature, and density increase across the shock and the flow velocity decreases [5]. The flow around a blunt body in a supersonic scenario is rotational because the shock wave is curved and each streamline experiences a different increase in entropy when crossing the shock. This results in an entropy gradient which causes the adiabatic flow behind the shock wave to be rotational [4].

---

\(^{10}\)In terms of ‘bluntness’ the flat plate completely obstructs the flow
Figure 18: Local approximation for normal shock wave in blunt body flow [4].

**Governing Equations** The Rankine-Hugoniot equations describe the discontinuous change in flow properties across normal shock waves and are given below [5, 28]^{11}

\[
M_2^2 = \frac{1 + ((\gamma - 1)/2)M_1^2}{\gamma M_1^2 - (\gamma - 1)/2} \quad (6)
\]

\[
\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \quad (7)
\]

\[
P_2 \frac{P_1}{P_1} = 1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1) \quad (8)
\]

\[
\frac{T_2}{T_1} = \frac{h_2}{h_1} = \frac{P_2}{P_1} \frac{\rho_1}{\rho_2} \quad (9)
\]

\[
s_2 - s_1 = c_p \ln(1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1))(\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}) - R \ln(1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1)) \quad (10)
\]

\[
T_{02} = T_{01} \quad (11)
\]

\[
P_{02} \frac{P_01}{P_01} = \exp^{-\frac{(s_2-s_1)}{R}} \quad (12)
\]

\[
e_2 - e_1 = \frac{P_1 + P_2}{2} (\nu_1 - \nu_2) \quad (13)
\]

^{11}state 1 corresponds to ahead of the shock, state 2 behind
4.3.1 Comparison to Euler 3-D Results

The results of the exact shock relations given by the Rankine-Hugoniot equations (6-13) are compared to the results of a UTIAS Euler 3-dimensional simulation. A large semi-vertex angle cone $\theta_c = 50^\circ$ was subjected to a $M = 2.0$ flow which resulted in a detached shock wave, see figure 19(a). The portion of the shock wave near the cone’s tip can be approximated as a 1-D normal shock, see figure 19(b), the analysis of this flow region simulates the current discussion of plate flow. The simulation was performed using the grid size $8 \times 8 \times 8$ because larger grids were found to not converge even after a large number of iterations, this phenomenon is discussed in greater detail in section 4.3.2. For a full discussion on the cone mesh see section 4.5.

![Figure 19](image)

(a) Sonic cone of detached shock. (b) Side profile of detached shock.

**Figure 19:** large semi-vertex angle cone $\theta_c = 50^\circ$ in $M = 2.0$ flow resulting in a detached shock wave.

The normal shock wave was numerically analyzed by importing the Euler 3-D data into Matlab®. The data in both blocks 1 and 2 was analyzed because the detached shock extends upstream from the cone. Since the flow is axially symmetric the Mach numbers, temperatures, and pressures within a $8 \times 8 \times 1$ grid slice were mapped to a square domain with the use of cell indices $(i, j)$ which simplifies the data visualization. The mapping and variable distributions are shown in figures 20(a), 20(c), 20(d), 20(f), 20(e) with the changes across the shock wave clearly shown. To emphasize the effects of the shock wave the derivative of the Mach number was taken and plotted using the same mapping, results are shown in figure 20(b). The strength of the approximate normal shock is clearly shown by the large Mach gradients near the tip of the cone.

The local results immediately after the normal shock wave were compared to the analytical results as given by equations 6-13. The upstream conditions as given by the Euler 3-D analysis were taken as state 1 and the downstream conditions, immediately after the normal shock were taken as state 2.
The downstream conditions were found by finding the largest gradient in each flow property and taking the value immediately after the location of the maximum gradient. The results of the comparison are summarized in table 1. The error in the results was found to be reasonably large, this is attributed to the coarse grid mesh size. Because there are only 2-3 cells immediately after the normal shock evaluation of the post-shock conditions are approximate.

**Table 1:** Comparison between Euler 3-D and analytic results for plate flow.

<table>
<thead>
<tr>
<th>Property</th>
<th>Euler 3-D</th>
<th>Analytic</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_2$</td>
<td>0.5148</td>
<td>0.5774</td>
<td>10.8417%</td>
</tr>
<tr>
<td>$\rho_2/\rho_1$</td>
<td>2.8304</td>
<td>2.6667</td>
<td>6.1387%</td>
</tr>
<tr>
<td>$P_2/P_1$</td>
<td>5.0560</td>
<td>4.5000</td>
<td>12.3556%</td>
</tr>
<tr>
<td>$T_2/T_1$</td>
<td>1.4888</td>
<td>1.6875</td>
<td>11.7750%</td>
</tr>
<tr>
<td>$s_2/s_1$</td>
<td>1.0164</td>
<td>1.0509</td>
<td>3.2910%</td>
</tr>
</tbody>
</table>

### 4.3.2 Flowfield Convergence Issues

As mentioned earlier convergence of the normal shock flow case was only achieved for a very coarse grid mesh. Finer meshes using a blunt 2-D wedge (see figure 75 in appendix C.2), a blunt 3-D cone (discussed previously) and several other blunt geometries shown in figure 21 were not able to converge to an acceptable error even though the general flow field was found to be qualitatively correct. It is suggested in [5, 4] that to solve the blunt body problem a time-dependant approach should be taken because of large regions of both supersonic and subsonic flow. The fact that the nature of the governing equations change from elliptic to hyperbolic across the sonic lines cause severe mathematical and numerical difficulties [5]. With this in mind both a time-dependant and a non-time dependant approach were attempted with similar results.

Possibilities for the convergence issues are the flow being rotational and/or unsteady. Because the blunt-body problem has a curved shock the flow behind the shock is rotational which is much more complicated than the irrotational cases found in wedge flow and cone flow presented in the next sections [4]. Furthermore the blunt body problem has large regions of subsonic flow which may cause the flow situation to be unsteady during the solution process.

It is acknowledged that a more in-depth analysis and investigation is required to solve the convergence issues for the plate-flow flow case but due to time limitations the current research has not attempted to isolate, understand, and solve the convergence issues associated with normal shock waves in blunt body flow.
Figure 20: Analysis of detached shock wave showing variable distributions over grid.
Figure 21: Un-converged supersonic flow at $M_\infty = 2.0$ past 2-D blunt bodies.
4.4 Wedge Flow

Wedge flow can be used to approximate the forward edges of supersonic aircraft wings and some supersonic engine inlet ramps [31]. An equivalent flow situation is the flow into a convex corner, aptly named a ‘compression corner’ [5]. Wedge flow is characterized by the formation of oblique shock waves with the flowfield being inherently two-dimensional. Oblique shocks occur when the supersonic flow is ‘turned into itself’ and the flow downstream is uniform and parallel, following the direction of the obstruction [5]. Contrary to normal shock waves $M_2$ behind an oblique shock need not be subsonic but $P_2, T_2, \rho_2$ are still increased and remain constant. Experimental images of supersonic flow past wedges in practical applications is given in figures 22(a), 22(b).

Figure 22: Interferogram and Schlieren images of supersonic flow past wedges.

Because oblique shock waves are straight all streamlines that pass through an oblique shock experience the same increase in entropy and therefore the flowfield behind an oblique shock is irrotational [5]. The wedge flow and resultant oblique shock waves in the current analysis are associated with a steady flow situation.

Governing Equations   The equations describing supersonic wedge flow were developed by Meyer [32] and are identical in form to the Rankine-Hugoniot relations (6-13) but with all $M$ replaced with the normal component $M_n$. This is because the changes across an oblique shock wave are governed by the normal component of the velocity only while the tangential component is preserved [5]. Three additional equations relating $M$ to $M_n$ and the $\theta - \beta - M$ relation\(^\text{12}\) completes the system [5, 28].

$$M_{n1} = M_1 \sin \beta$$

\(^{12}\text{which specifies } \theta \text{ as a unique function of } M_1 \text{ and } \beta\)
\[ M_2 = \frac{M_{n2}}{\sin(\beta - \theta)} \]  
(15)

\[ \tan \theta = 2 \cot \beta \left( \frac{M_2^2 \sin^2 \beta - 1}{M_1^2 (\gamma + \cos 2\beta) + 2} \right) \]  
(16)

Figure 23: Oblique shock \( \theta - \beta - M \) relationships with sonic lines shown.

**Primary Numerical Results**  Solutions to equation 16 were computed numerically using Matlab® and are presented in figure 23(a) with comparison to the results presented in [5] in figure 23(b). The current numerical results agree with the results of [5] to within the accuracy the results of [5] can be measured\(^{13}\). For a given \( M_1 \) there is a max wedge semi-vertex angle \( \theta_{\text{max}} \) for which a straight oblique shock solution exists. If \( \theta > \theta_{\text{max}} \) then a curved detached shock occurs with no given solution. The maximum wedge semi-vertex angle \( \theta_{\text{max}} \) increases with increasing \( M_1 \). There is a large increase in \( \theta_{\text{max}} \) from \( M_1 = 2 \) to \( M_1 = 3 \) with the increases in successive \( M_1 \) diminishing at higher \( M_1 \). The maximum possible wedge semi-vertex angle was found numerically by setting \( M_\infty = 10^{100} \approx \infty \) with a result of \( \theta_{\text{max}} = 45.5847^\circ \) at a shock wave angle of \( \beta = 67.792^\circ \). This value is validated by comparing to the stated \( \theta_{\text{max}} = 45.6^\circ \) of [28]. A wedge semi-vertex angle larger than this value will always result in a detached shock wave and no oblique shock solution.

Both strong and weak shock solutions exist with the weak solutions given by the curves below the sonic line in figure 23(a). The weak shock usually occurs in nature and therefore only the sonic line for the weak shock solutions were calculated.

\(^{13}\)due to the poor reproduction quality of the figure
4.4.1 Further Discussion and Analysis of Wedge Flow

Another useful tool for analyzing oblique shock waves in wedge flow is the shock polar, first introduced by Meyer in his dissertation [32]. A shock polar is a diagram of the resultant velocity components \( V_{x2} \) and \( V_{y2} \) of the deflected flow after passing through an oblique shock wave given an initial flow of \( V_{x1} \). The geometry for the given situation is presented in figure 76(a) in appendix C.3. We nondimensionalize the \( V \) components using \( a^* \) resulting in expressions with characteristic Mach numbers \( M^* \). The quantitative behavior of \( M^* \) is the same as \( M \) however \( M^* \) approaches a finite number as \( M \to \infty \), allowing us to graph the shock polar nicely in one diagram [33]. The expression for the shock polar has been derived from the geometry\(^{16}\) of figure 76(b) in appendix C.3 and is given as

\[
(V_y/a^*)^2 = \frac{2}{\gamma+1} \left( \frac{V_{x1}}{a^*} \right)^2 - \frac{\gamma}{\gamma+1} \left( M^* \right)^2 \left( \frac{V_{x1}}{a^*} \right) M^* + \frac{\gamma}{\gamma+1} (17)
\]

The shock polar for wedge flow was numerically calculated in Matlab\textsuperscript{®} using equation 17. The results are given in figure 24(a) along with useful diagram for interpreting the results given in figure 24(b). In figure 24(b) points \( B, D \) represent the weak and strong shock solutions for a given \( \theta \), the tangent line \( OC \) is the maximum deflection angle \( \theta_{max} \), point \( E \) is the normal shock solution, point \( A \) is the Mach line, and \( \beta \) is the wave angle. Inside the *sonic circle* all velocities are subsonic, outside it all velocities are supersonic [33, 5]. Another useful variation of the shock polar is Meyer’s pressure ratio polar given in figure 77 in appendix C.3 which relates \( P_1/P_0 \) before the shock to \( P_2/P_0 \) after the shock with \( \theta \) as a parameter.

![Oblique Shock Polar](image)

(a) Numerical shock polar for \( M_1 = 2 - 5 \).

![Shock Polar Diagram](image)

(b) Physical interpretation of shock polar [5].

**Figure 24:** Numerical shock polar and interpretation for wedge flow.

\(^{14}\)initial velocity parallel to x-axis so \( V_{y1} = 0 \)

\(^{15}\)starred quantities are achieved by adiabatically slowing down fluid to \( M = 1 \)

\(^{16}\)following the derivation of [33]
4.4.2 Comparison to Euler 3-D Results

The results of the exact shock relations given by the modified Rankine-Hugoniot equations (6-16) are compared to the results of a UTIAS Euler 3-dimensional simulation. A finite half-wedge was modeled in three dimensions, the basic grid outline and block structure is given in figure 78 in appendix C.3.

Effects of Grid Size   Before a full investigation of supersonic wedge flow was undertaken a preliminary analysis on the effects of the grid size was performed. The grid initially contains two blocks with the shock wave occurring in the second block. A sample wedge flow case with \( M_\infty = 3 \) and wedge semi-vertex angle \( \theta = 15^\circ \) was run with block grids of \( 4 \times 4 \times 4 \), \( 8 \times 8 \times 4 \), and \( 16 \times 16 \times 4 \). The results are shown in figures 25(a), 25(b), 25(c) and reveal that increasing the grid size dramatically sharpens the shock wave location. With a grid size of \( 4 \times 4 \times 4 \) the shock wave region is very large, curved, and not well defined. As the grid size is increased to \( 8 \times 8 \times 4 \) the shock wave region decreases in size, curvature, and becomes more defined. This trend continues to the grid size \( 16 \times 16 \times 4 \) however it is not until we increase the grid size to \( 16 \times 16 \times 4 \) that the shock wave region has a reasonable well defined shock angle. Increasing the grid size above \( 16 \times 16 \times 4 \) resulted in a significant increase in computational time since only the frontend of HPACF was used. It was also found that the data manipulation time using Tecplot® and Matlab® was significantly increased for finer grid meshes and would become a serious bottleneck in the analysis cycle. Since the primary purpose of the analysis was to compare the Euler 3-D results for a wide range of \( M_\infty \) and wedge semi-vertex angles \( \theta \) rather than focus on a single configuration it was decided that the \( 16 \times 16 \times 4 \) grid was an adequately sized mesh with an acceptable balance between accuracy, computational time, and data manipulation time.

Incorporation of AMR   The Euler 3-D code has the ability to incorporate Automatic-Mesh-Refinement (AMR) to a maximum of two levels of refinement. The effects of AMR on the wedge flow case are shown in figure 26. AMR located the regions of large gradients due to the oblique shock wave and refined the mesh in the appropriate areas. Because AMR only allows 2-1 ratio cell boundaries between blocks the region ahead of the oblique shock is also refined when two levels of AMR refinement are applied. For the wedge flow case AMR was only investigated briefly but was not used for the bulk analysis, this was due to fact that the data manipulation and analysis was written for individual blocks. With uniform refinement a single block contained the entire oblique shock but with AMR the oblique shock was divided into 16+ blocks and would have been difficult to analyze with the current analysis implementation. 

Shock Wave Angles   The angles of the shock waves were numerically computed by importing the Euler 3-D data into Matlab®. Only the data in the second block containing the shock wave was analyzed. The Mach numbers within the \( 16 \times 16 \times 1 \) grid were mapped to a square domain with the use of cell indices \( (i, j) \) that are then mapped to distances \( (x, y) \) which simplifies the data visualization. The mapping and \( M \) distribution are shown for a sample run in figures 27(a), 27(b) with the change in Mach number across the

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17the dimensions along the \( i-j-k \) or \( x-y-z \) directions respectively
18the \( 16 \times 16 \times 4 \) grid approx 90s computation time and the \( 32 \times 32 \times 4 \) grid approx 300s computation time
19the limit on refinement levels is currently not understood with reference to wedge flow [26]
(a) Numerical results for $M = 3.0$, $\theta = 15^\circ$, Grid size $4 \times 4 \times 4$.

(b) Numerical results for $M = 3.0$, $\theta = 15^\circ$, Grid size $8 \times 8 \times 4$.

(c) Numerical results for $M = 3.0$, $\theta = 15^\circ$, Grid size $16 \times 16 \times 4$.

Figure 25: Effects of block grid size in wedge flow analysis.
Figure 26: Automatic-Mesh-Refinement (AMR) application to wedge flow.
shock wave clearly emphasized. To emphasize the effects of the shock wave the derivative of the Mach number is taken and plotted using the same mapping, sample results are shown in figure 27(c). To find the shock wave angle the cells where the derivative becomes non-zero are found then the original wedge angle is added and the data is plotted, a least squares fit is then applied to the data and the shock wave angle is extracted, see figure 27(d). Another method was attempted in which the maximum change in $M$ within the shock wave was plotted and the shock wave angle extracted from a least squares fit, however this method resulted in poorer data correlation and was determined not applicable for the $16 \times 16 \times 4$ mesh size. The results from the analysis are compared to the analytical results and are given in figure 28. Because of the significant error in some flow situations the effect of increasing the grid size on the determination of the shockwave angle a grid size of $32 \times 32 \times 4$ was investigated for two flow cases shown in figure 79 in appendix C.3, with the results given in table 2. It was found that the refinement decreased the error in the calculated shock wave angle by an order of magnitude in both cases. This shows that grid refinement leads to a more accurate solution which in the limit would converge to the exact solution, this further confirms the validity of the current results despite the significant errors in the original analysis.

Table 2: Effect of further grid refinement on wedge shockwave angle.

<table>
<thead>
<tr>
<th>grid size</th>
<th>$M$</th>
<th>$\theta$</th>
<th>$\beta$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16 \times 16 \times 4$</td>
<td>4</td>
<td>$15^\circ$</td>
<td>25.43°</td>
<td>6.02%</td>
</tr>
<tr>
<td>$32 \times 32 \times 4$</td>
<td>4</td>
<td>$15^\circ$</td>
<td>26.84°</td>
<td>0.813%</td>
</tr>
<tr>
<td>$16 \times 16 \times 4$</td>
<td>5</td>
<td>$10^\circ$</td>
<td>19.40°</td>
<td>0.155%</td>
</tr>
<tr>
<td>$32 \times 32 \times 4$</td>
<td>5</td>
<td>$10^\circ$</td>
<td>19.38°</td>
<td>0.0516%</td>
</tr>
</tbody>
</table>

Shock Polar  The shock polar was computed by finding the relation between the adiabatically normalized $u$ and $v$ velocities. The variation of these velocities across the shock wave for the $M = 5.0, \theta = 30^\circ$ case is given in figures 29(a), 29(b). The $u$ velocity is decreased across the shock wave to some non-zero value and the $v$ velocity is increased from zero to some non-zero value, results that are consistent with the definition of an oblique shock wave. The shock polar was calculated over a range of $M$ and $\theta$ and is compared to the analytic results in figures 29(c), 29(d). The simulated Euler 3-D results were found to generally overpredict $V_y/a^*$ for a given $V_x/a^*$. 
(a) Grid $16 \times 16 \times 4$ with second block $(i,j)$ indices.

(b) Mach distribution over second block grid.

(c) Mach derivative distribution over second block grid.

(d) $(i,j)$ locations of shock wave-front with least squares fit.

**Figure 27:** Analysis of shock wave angle, $M = 3.0$, $\theta = 20^\circ$. 
Figure 28: Comparison between Euler 3-D and analytically computed shock angles.
(a) Variation in $u/a^*$ across shockwave.
(b) Variation in $v/a^*$ across shockwave.
(c) Euler 3-D and analytic shock polar comparison.
(d) Percent error of calculated shock polar, negative indicating an overprediction in $V_y/a^*$.

**Figure 29:** Calculation of shock polar with Euler 3-D for $M = 5.0, \theta = 30^\circ$. 
4.5 Conical Flow

Conical flow is characterized by the formation of 3-D, axisymmetric oblique shock waves. The analysis of conical flow is of great practical importance; many high-speed missiles, supersonic aircraft, and projectiles have fuselage or nacelle nosecones that are approximately conical [5, 6]. The importance of the flow pattern is not strictly limited to cones, the solutions of conical flow are applicable to the tip regions of any sharp-nosed body of revolution, for which one can imagine a wide variety of applications are possible [6]. Furthermore the development of the numerical solution for conical flow is a hallmark in the development of compressible flow and its applications, it represents a significant historical achievement that has led the way to more developed analysis of supersonic flow geometries [5]. Schlieren images of supersonic flow past cones in practical applications is given in figures 30(a), 30(b).

![Schlieren images of supersonic flow past cones.](image)

If the wedge of Meyer’s analysis is replaced with an axisymmetric cone, a similar solution does not exist. This occurs because the air cannot travel in the direction into which it was first deflected once it has passed through the oblique shock wave [27]. Indeed the flow properties immediately behind the oblique shock are expressed using the familiar oblique shock relations. However because the flow over a cone is three-dimensional the flowfield between the shock and the surface of the cone is not uniform. The streamlines are curved and have extra space to move through hence ‘relieving some of the obstructions set up by the presence of the body’ [5]. It was found by Taylor and Maccoll [20] that an irrotational solution of the flow between the oblique shock and the cone predicts constant flow properties over coaxial cones passing through the vertex of the cone. The suggestion for such a solution was first presented by Busemann [21] who gave a graphical method for obtaining them. The conical equations are derived directly from the governing equations of fluid flow, however the notation used by Taylor-Maccoll is somewhat antiquated and lacks the insight presented through the proper use of variable definitions, so we will use the notation presented in more recent compressible flow texts such as [5].

---

20 the work of Taylor-Maccoll was inspired by the approximate solutions presented by v. Karman and Moore [27].
21 a discussion of Busemann’s apple curves is given in [6] with an example given in figure 88 in appendix C.4.
As a preview the comparison of calculated surface pressures with wind tunnel observations as well as shock wave angles from [27] is given in figures 80, 81 in appendix C.4. It can be seen that the solution of Taylor-Maccoll accurately predicts both the surface pressure distributions and wave angles when considering the possible experimental errors of the 1930’s.

4.5.1 Derivation of Conical Flow Equations: Taylor and Maccoll Solution

The axisymmetric supersonic flow over a sharp cone at zero angle of attack is nonlinear, however it is a special degenerate case of three-dimensional flow so we can solve it exactly by following the work of Taylor-Maccoll [5]. Using cylindrical coordinates the flowfield is symmetric about the z-axis which is aligned with the axis of symmetry, so all properties are independent of $\phi$ and the flowfield is only dependant on $r$ and $z$. Because the flow takes place in three-dimensional space but only has two independent variables it is termed ‘quasi-two-dimensional’ [5]. Before we begin the quantitative solution we first examine the physical nature of the problem. A sharp cone of semi-vertex angle $\theta_c$ extends infinitely within a steady supersonic flow. A straight oblique shock wave is attached to vertex of the cone in which the flow streamlines deflect discontinuously and then continue to curve continuously downstream of the shock, becoming parallel to the cones surface at infinity. Because there is no meaningful scale length for the semi-infinite cone the flow properties along rays of constant $\theta$ from the vertex must be constant, indeed this is definition of conical flow [5]. Because all streamlines have the same entropy change across the shockwave so we know that the flow behind the shock is isentropic and irrotational [6]. The described flow geometry along with the proper nomenclature is given in figures 31(a), 31(b).

Governing Equations  The conservation of mass, condition of irrotationality, and Euler equation are used in this analysis but are significantly simplified due to the steady axisymmetric conical flow conditions.

---

22 for example the use of mercury columns balanced against atmospheric pressure [35]
23 this is because the flow is steady and the oblique shock wave is straight, furthermore the stagnation enthalpy and temperature are constant [36]
In spherical coordinates they are

\[ 2V_r + V_\theta \cot \theta + \frac{dV_\theta}{d\theta} + \frac{V_\theta d\rho}{\rho d\theta} = 0 \]  
(18)

\[ \frac{d\rho}{d\theta} = -\frac{2\rho}{\gamma - 1} \left( V_r \frac{dV_r}{d\theta} + V_\theta \frac{dV_\theta}{d\theta} \right) \]  
(19)

\[ V_\theta = \frac{dV_r}{d\theta} \]  
(20)

\[ V^2 = V_r^2 + V_\theta^2 \]  
(21)

Substituting equation 19 into equation 18 and using equation 20 we derive the Taylor-Maccoll equation for the solution of conical flows.

\[ \frac{\gamma - 1}{2} \left( V_{\text{max}}^2 - V_r^2 - \left( \frac{dV_r}{d\theta} \right)^2 \right) (2V_r + \frac{dV_r}{d\theta} \cot \theta + \frac{d^2V_r}{d\theta^2}) - \frac{dV_r}{d\theta} \left( V_r \frac{dV_r}{d\theta} + \frac{dV_r}{d\theta} \left( \frac{d^2V_r}{d\theta^2} \right) \right) = 0 \]  
(22)

The Taylor-Maccoll equation is a nonlinear second order ordinary differential equation with one independent variable \( V_r \) that applies from the shock \( \theta_s \) to the surface of the cone \( \theta_c \) [37, 6]. The solution of \( V_r = f(\theta) \) leads to the solution for \( V_\theta \) through equation 20. No analytic solution exists for equation 22 and therefore it must be solved numerically. To simplify the numerics we nondimensionalize equation 22 using the nondimensional velocity \( V' \) which is defined as

\[ \frac{V}{V_{\text{max}}} \triangleq V' = \left( \frac{2}{(\gamma - 1)M^2} + 1 \right)^{-1/2} \]  
(23)

We can see that \( V' = f(M) \) from equation 23. We then transform the Taylor-Maccoll equation 22 to its nondimensional form.

\[ \frac{\gamma - 1}{2} \left( 1 - V_r'^2 - \left( \frac{dV_r'}{d\theta} \right)^2 \right) (2V_r' + \frac{dV_r'}{d\theta} \cot \theta + \frac{d^2V_r'}{d\theta^2}) - \frac{dV_r'}{d\theta} \left( V_r' \frac{dV_r'}{d\theta} + \frac{dV_r'}{d\theta} \left( \frac{d^2V_r'}{d\theta^2} \right) \right) = 0 \]  
(24)

**Numerical Procedure** An inverse approach is taken to the solution of this problem where the flow Mach number \( M_\infty \) and the shock angle \( \theta_s \) are assumed. We construct a numerical solver using the ode15s function that is available in Matlab to solve the nondimensional Taylor-Maccoll ODE for \( V_r(\theta) \). The boundary conditions for \( V_r \) and \( V_\theta \) are determined immediately behind the shock using the \( \theta - \beta - M \) oblique shock relations. The other boundary condition is the flow tangency condition at the surface of the solid cone [37]. Both boundary conditions as well as the flow properties for the solution are shown in figure 32. We solve for \( V_r(\theta) \) by marching the solution from the shock wave \( \theta_s \) until \( \theta_c \) is reached where

---

24 this is true for a calorically perfect gas, which is assumed in the current analysis  
25 a technique that is presented by Anderson [5]  
26 the procedure is similar to that of Lassaline [37]
we know that $V_\theta = 0$ is a boundary condition. This uniquely defines the cone semi-vertex angle $\theta_c$ for the initial $M_\infty$ and $\theta_s$ conditions. Further solutions are obtained by varying these parameters and solving for $\theta_c$.

![Conical flow properties and boundary conditions](image)

**Figure 32:** Conical flow properties and boundary conditions for numerical solution [37].

To effectively use Matlab’s® ode15s function to solve the nondimensional Taylor-Maccoll equations 24 it is convenient to represent the ODE as a system of first order ODE’s with the solution vector being

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_r \\ V_\theta \end{bmatrix}$$

(25)

so that we can then represent the system of first order ODE’s by rearranging equation 24 for $\frac{d^2V_r}{d\theta^2}$ with the following vector

$$x' = \begin{bmatrix} V_\theta \\ \frac{V^2_r V_\theta - \frac{\gamma-1}{2} (1-V^2_r-V^2_\theta) (2V_r+V_\theta \cot \theta)}{\frac{\gamma}{2} (1-V^2_r-V^2_\theta) - V_\theta^2} \end{bmatrix}$$

(26)

The entire flow between the shock wave and the cone surface is found using equations 25 and 26, the local $M$ can be found using equation 21 and the remaining flow properties can be found using the isentropic flow relations

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

(27)

$$\frac{P_0}{P} = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{\gamma}{\gamma-1}}$$

(28)

$$\frac{\rho_0}{\rho} = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{1}{\gamma-1}}$$

(29)
Primary Numerical Results for Analytical Solution  A number of properties were investigated for various $M_\infty$. The shock wave angle $\theta_s$, cone surface Mach $M_c$, static pressure ratio $\frac{P_s}{P_1}$, and stagnation pressure ratio $\frac{P_0}{P_01}$ were calculated as a function of the cone semi-vertex angle $\theta_c$, with the results shown in figure 33. The conical flow solutions reveal that for a given cone angle $\theta_c$ and given $M_\infty$ there are two possible solutions, the strong and weak shock solutions. In each of the figures two possible solutions exist for all permission cone semi-vertex angles $\theta_c$. The weak solution is generally observed on real finite cones but it is possible to achieve the strong solution by increasing the cone back pressure [5]. The results of the current analysis was compared to tabulations of results given in NASA report SP-3004 [38]\(^{27}\). In all cases the solutions agree to as many significant figures as are presented.

(a) Shock wave angle given semi-vertex and Mach.
(b) Cone surface Mach given semi-vertex and Mach.
(c) Static pressure ratio given semi-vertex and Mach.
(d) Stagnation pressure ratio given semi-vertex and Mach.

Figure 33: Numerical results for conical flow using Taylor-Maccoll equations.

\(^{27}\)which are essentially identical to Kopal’s results [20].
4.5.2 Further Discussion and Analysis of Conical Flow

It was noticed by [27] that the flow behind the conical shock may be completely supersonic, subsonic, or some mixture with the vertex angle $\theta_a$ which is intermediate between $\theta_w$ and $\theta_s$ denoting the transition from supersonic to subsonic. The values of $\theta_a$ for a variety of cone flow conditions are given in Table 2 of [27]. In this case one of the $\theta$ rays from the cone vertex becomes a sonic line as shown in figure 34. This is one of the few instances in nature where supersonic flow is isentropically compressed from supersonic to subsonic velocities\(^{28}\) [5]. It was also noted by [27] that stationary wavelets can exist in the flow at the local mach angle,

$$\theta_{\text{wave}} = f(M_{\text{cone}}) = \arcsin\left(\frac{1}{M_{\text{cone}}}\right)$$  \hspace{1cm} (30)

These wavelets are only likely to occur for the cases when supersonic flow extends to the cone-boundary where irregularities on the surface of the solid cone would result in disturbances. A diagram depicting these stationary wavelets is given in figure 35.

Sonic Lines from Numerical Analysis The conditions for the existence of isentropic compression of the flow with the presence of a well defined sonic line was investigated. It was found that only a limited number of cone semi-vertex angles $\theta_c$ resulted in isentropic compression of the flowfield from supersonic to subsonic velocities\(^{29}\). This is one of the few instances in nature where a supersonic flowfield is isentropically compressed to a subsonic velocity. A transition from supersonic to subsonic velocities is usually accompanied by shock waves, however this is a special exception to that generalization [5]. The results are summarized in table 3 for a Mach range of $M_\infty = 2 - 5$. For cone semi-vertex angles larger than those presented the flow was completely subsonic behind the oblique shock wave, while for cone semi-vertex angles lower than those presented the flow was completely supersonic behind the oblique shock wave. The 'window' of valid cone semi-vertex angles was found to decrease with increasing $M_\infty$. The analysis was conducted over a fairly coarse grid with the cone semi-vertex angle $\theta_c$ being varied\(^{28}\) usually a transition from supersonic to subsonic velocities is accompanied by shock waves which increase the flow entropy\(^{29}\) note that the more general case of isentropic compression without transition always occurs in cone flow.

---

\(^{28}\) usually a transition from supersonic to subsonic velocities is accompanied by shock waves which increase the flow entropy

\(^{29}\) note that the more general case of isentropic compression without transition always occurs in cone flow.
by $1.0^\circ$, therefore the results presented in table 3 do not cover the entire possible range. Examples of the flow geometry for the lower and upper $M_\infty$ range are presented in figures 42(d), 36(b). The flow is supersonic immediately behind the oblique shock wave but is isentropically compressed, passing through a well defined sonic line\(^{30}\), and becomes subsonic at the cones surface $\theta_c$. It was found that the location of the sonic line $\theta_{\text{sonic}}$ approaches the cones surface $\theta_c$ for higher $M_\infty$.

\[^{30}\text{defined as the } \theta \text{ ray from the cone vertex for which } M = 1\]

\(\begin{align*}
(a) \quad & M = 2, \theta_s = 60.0^\circ, \theta_c = 38.0676^\circ, \theta_{\text{sonic}} = 54.8233^\circ. \\
(b) \quad & M = 5, \theta_s = 65.0^\circ, \theta_c = 52.2477^\circ, \theta_{\text{sonic}} = 56.7632^\circ.
\end{align*}\)

**Figure 36:** Numerical results for isentropic compression with transition in conical flow.

It was found by [27] that a critical value for the incoming airflow existed which depended on the cone semi-vertex angle. Flow at a higher velocity than the critical produced attached conical flow which can be analyzed by the governing equations, however flow at a lower velocity than the critical resulted in a detached shock wave for which the equations presented are no longer applicable. Example values of the flow geometry for the lower and upper $M_\infty$ range are presented in figures 42(d), 36(b). The flow is supersonic immediately behind the oblique shock wave but is isentropically compressed, passing through a well defined sonic line, and becomes subsonic at the cones surface $\theta_c$. It was found that the location of the sonic line $\theta_{\text{sonic}}$ approaches the cones surface $\theta_c$ for higher $M_\infty$.

\[^{30}\text{defined as the } \theta \text{ ray from the cone vertex for which } M = 1\]
Table 3: Table of results for analytic isentropic compression with transition in conical flow.

<table>
<thead>
<tr>
<th>$M_\infty$</th>
<th>$\theta_s$</th>
<th>$\theta_c$</th>
<th>$\theta_{sonic}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>58.0000°</td>
<td>37.0033°</td>
<td>44.1572°</td>
</tr>
<tr>
<td>2</td>
<td>59.0000°</td>
<td>37.5527°</td>
<td>50.6175°</td>
</tr>
<tr>
<td>2</td>
<td>60.0000°</td>
<td>38.0676°</td>
<td>54.8233°</td>
</tr>
<tr>
<td>2</td>
<td>61.0000°</td>
<td>38.5458°</td>
<td>59.2960°</td>
</tr>
<tr>
<td>3</td>
<td>62.0000°</td>
<td>46.5870°</td>
<td>52.0155°</td>
</tr>
<tr>
<td>3</td>
<td>63.0000°</td>
<td>47.0814°</td>
<td>58.9700°</td>
</tr>
<tr>
<td>4</td>
<td>64.0000°</td>
<td>50.4050°</td>
<td>55.6968°</td>
</tr>
<tr>
<td>4</td>
<td>65.0000°</td>
<td>50.8754°</td>
<td>63.3449°</td>
</tr>
<tr>
<td>5</td>
<td>65.0000°</td>
<td>52.2477°</td>
<td>56.7632°</td>
</tr>
<tr>
<td>5</td>
<td>66.0000°</td>
<td>52.7151°</td>
<td>65.5614°</td>
</tr>
</tbody>
</table>

this critical velocity were calculated using graphical interpolation [27] and validated by bullet shadowgraphs [35] and are given in table 4. According to the governing equations the pressure at the surface of the cone is uniform for conical flow [35]. Experiments performed on a 30° semi-vertex cone below the critical velocity showed considerable decrease in pressure along the surface of the cone with increasing distance from the vertex [35]. Diagrams depicting the phenomena of attached and detached conical flow are given in figure 37 with experimental bullet observations given in figures 82, 83 in appendix C.4.

Table 4: Examples of critical velocity for conical flow from bullet shadowgraphs [35].

<table>
<thead>
<tr>
<th>$\theta_c$</th>
<th>$M_{crit}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>1.46</td>
</tr>
<tr>
<td>20°</td>
<td>1.18</td>
</tr>
<tr>
<td>10°</td>
<td>1.035</td>
</tr>
</tbody>
</table>

During bullet shadowgraph experiments performed by the Research Department of Woolwich Arsenal [35] it was found that the nature of the flow was affected by the finite cone (in comparison to the infinite cone for which the equations apply) for a range of flow velocities. For the finite bullet the change at the shoulder from the conical head to the parallel middle body can affect the shape of the shock wave for cases where subsonic flow exists between the shock wave and the bullet surface. For large enough velocities the flow between the shock and the bullets surface will be entirely supersonic and the disturbance caused by the shoulder cannot propagate into this region so that the shock wave should be truly conical. For the range of velocities with attached conical flow but regions of subsonic between the shock and bullets surface this disturbance affects the upstream flow\textsuperscript{31}, with a consequent increase in the shock wave curvature near the conical portion of the bullet [35].

The critical velocities for attached flow for the supersonic wedge and supersonic cone cases discussed previously have been calculated by Taylor-Maccoll and are presented in figures 85(a), 85(b), 86 in appendix C.4. It is shown that as the velocity is decreased or the semi-vertex angle increased the shock

\textsuperscript{31}this range being $1.46 < M < 1.66$ for the 30° semi-vertex bullet [35]
wave leaves the wedge long before the cone. The cone experiences a ‘three-dimensional relieving effect’ and therefore experiences a lower surface pressure, temperature, density, entropy, and shock angle when compared to the supersonic wedge case [5]. For cases where the cone semi-vertex angle becomes too large \( \theta_c > \theta_{c_{\text{max}}} \) there exists no Taylor-Maccoll solution and the detached shock flowfield must be solved using time-stepping numerical techniques.

**Critical Velocities from Numerical Analysis**  The critical velocities for attached conical flow\(^{32}\) have been investigated using the present numerical analysis with the results shown in figure 38. The results are identical to those presented in figure 85(a) but extends the analysis up to \( M_\infty = 5 \). The maximum possible deflection \( \theta_{\text{max}} \) for any \( M_\infty \) was found by setting \( M_\infty = 10^{100} \approx \infty \) with the resultant \( \theta_{\text{max}} = 57.6862^\circ \) which agrees with the stated value of \( \theta_{\text{max}} = 57.5^\circ \) in [28]. A cone semi-vertex angle larger than this value will always result in a detached shock wave and no Taylor-Maccoll solution.

**Alternative Solution Methods**  In the discussion of the Taylor-Maccoll solution for conical supersonic flow presented in [35] the approximate method for determining the disturbances produced by a thin spindle-shaped body moving at supersonics speeds of \( v \) Karman and Moore was presented and compared. The method assumes irrotational flow without shock waves. It is known that for a body in supersonic flow shock waves will be present, however the method assumes that the change in pressure at the shock wave is small when compared to the change in pressure between the shock wave and the solid surface [35]. For the case of a cone with a \( 10^\circ \) semi-vertex angle this pressure change ratio\(^{33}\) is given in figure 84 for a range of flow velocities. A comparison between the exact method of Taylor-Maccoll for the cone and the approximate method of \( v \) Karman and Moore was performed in [35] for the pressure on a cone with varying semi-vertex angle. The results are given in figure 87 in appendix C.4 and show that the first order approximation is valid for small angle cones over a range of velocities.

---

\(^{32}\)Corresponding to a Taylor-Maccoll solution

\(^{33}\)Denoted the variable \( F \)
Figure 38: Numerical results for the critical velocities for cone flow with $\theta_c$ in deg.

4.5.3 Comparison to Euler 3-D Results

The results of the exact shock relations given by the Taylor-Maccoll equations (equations 22-24) are compared to the results of a UTIAS Euler 3-D simulation. A finite half-cone was modeled in three dimensions, the basic grid outline and block structure is given in figure 89 in appendix C.4. To construct the cone geometry a ‘rotator’ function was created to rotate the 2-D cross section of the wedge flow geometry instead of extruding it as in the wedge flow case in section 4.4.2. To save computational resources and time the wedge geometry was only rotated through an angle $\pi$ to create a half-cone.

Effects of Grid Size  Before a full investigation of supersonic cone flow was undertaken a preliminary analysis on the effects of the grid size was performed. The grid initially contains two blocks representing half the full cone\(^{34}\) with the shock wave occurring in the second block. A sample cone flow case with $M_\infty = 3$ and wedge semi-vertex angle $\theta = 30^\circ$ was run with block grids of $4 \times 4 \times 4$, $8 \times 8 \times 8$, and $16 \times 16 \times 16$\(^{35}\). The results are shown in figure 39 and reveal that increasing the grid size dramatically sharpens the shock wave location as well as resulting in a much smoother cone geometry. With a grid size of $4 \times 4 \times 4$ the shock wave region is very large, curved, and not well defined. As the grid size is increased to $8 \times 8 \times 8$ the shock wave region decreases in size, curvature, and becomes more defined. This trend continues to the grid size $16 \times 16 \times 16$ however it is not until we increase the grid size to $16 \times 16 \times 16$ that the shock wave region has a reasonable well defined shock angle. In similarity to the wedge flow analysis increasing the grid size above $16 \times 16 \times 16$ resulted in a significant increase in the computational time\(^{36}\) as well as the data manipulation time. Due to the identical reasoning presented in the wedge flow discussion

\(^{34}\)since the flow case is axisymmetrical, simulating only half the cone saved computational time without decreasing accuracy

\(^{35}\)the dimensions along the $i-j-k$ or $x-y-z$ directions respectively

\(^{36}\)the $16 \times 16 \times 16$ grid approx 220s computation time and the $32 \times 32 \times 32$ grid approx 1500s computation time
(section 4.4.2) and the fact that the primary purpose of this analysis is to compare the Euler 3-D results for a wide range of $M_\infty$ and cone semi-vertex angles $\theta$ rather than focus on a single configuration the $16 \times 16 \times 16$ grid was chosen as an adequately sized mesh with an acceptable balance between accuracy, computational time, and data manipulation time.

**Incorporation of AMR**  The Euler 3-D code has the ability to incorporate Automatic-Mesh-Refinement (AMR) to a maximum of two levels of refinement$^{37}$. The effects of AMR are shown in figure 40. AMR located the regions of large gradients due to the oblique, axisymmetric shock wave and refined the mesh in the appropriate areas. Because AMR only allows 2-1 ratio cell boundaries between blocks the region ahead of the shock is also refined when two levels of AMR refinement are applied. For the cone flow case AMR was only investigated briefly but was not used for the bulk analysis, this was due to fact that the data manipulation and analysis was written for individual blocks. With uniform refinement a single block contained the entire oblique shock but with AMR the shock was divided into 8+ blocks and would have been difficult to analyze with the current analysis implementation.

**Shock Wave Angles**  The angles of the shock waves were numerically computed by importing the Euler 3-D data into Matlab$^\text{®}$. The analysis techniques are identical to those presented in the wedge flow analysis section 4.4.2 for shock wave angles. The results from the analysis are compared to the analytical results and are given in figure 41. Due to the reasonably significant errors in some cases a further refinement study was performed on two conical flow situations shown in figure 90 in appendix C.4 with the results presented in table 5. The error was found to decrease by a factor of $1/4$ to $1/2$ which is less than in the wedge flow analysis but still indicates that the solution is converging in the limit to the analytical solution.

**Table 5:** Effect of further grid refinement on cone shockwave angle.

<table>
<thead>
<tr>
<th>grid size</th>
<th>$M$</th>
<th>$\theta$</th>
<th>$\beta$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16 \times 16 \times 16$</td>
<td>2</td>
<td>10°</td>
<td>31.26°</td>
<td>0.160%</td>
</tr>
<tr>
<td>$32 \times 32 \times 32$</td>
<td>2</td>
<td>10°</td>
<td>31.25°</td>
<td>0.128%</td>
</tr>
<tr>
<td>$16 \times 16 \times 16$</td>
<td>3</td>
<td>30°</td>
<td>39.05°</td>
<td>1.835%</td>
</tr>
<tr>
<td>$32 \times 32 \times 32$</td>
<td>3</td>
<td>30°</td>
<td>39.35°</td>
<td>1.081%</td>
</tr>
</tbody>
</table>

**Sonic Lines from Euler 3-D**  A number of the flow situations which resulted in an isentropic flow compression from supersonic to subsonic Mach numbers as discussed previously and presented in table 3 were simulated and investigated using the Euler 3-D code. The angle of the sonic line was measured by taking a planar cut of the sonic cone in Matlab$^\text{®}$ then interpolating the points where the Mach number equaled one from the original data and then performing a linear fit. An example of the analysis and comparison to the analytic results is given in figure 42. A full list of the sonic line results from the Euler 3-D analysis is given in table 7 in appendix C.4.

$^{37}$the limit on refinement levels is currently not understood with reference to cone flow [26]
(a) Numerical results for $M = 3.0, \theta = 30^\circ$, Grid size $4 \times 4 \times 4$.

(b) Numerical results for $M = 3.0, \theta = 30^\circ$, Grid size $8 \times 8 \times 8$.

(c) Numerical results for $M = 3.0, \theta = 30^\circ$, Grid size $16 \times 16 \times 16$.

**Figure 39:** Effects of block grid size in cone flow analysis, only second block shown.
(a) Single level of AMR refinement, $M = 5.0$, $\theta = 30^\circ$.

(b) Two levels of AMR refinement, $M = 5.0$, $\theta = 30^\circ$.

**Figure 40:** Automatic-Mesh-Refinement (AMR) application to cone flow.
Figure 41: Comparison between Euler 3-D and analytically computed shock angles.
Figure 42: Isentropic compression analysis for $M = 2$, $\theta_c = 38.00^\circ$ conical flow, Grid size $16 \times 16 \times 16$. 

(c) Euler 3-D results, $\theta_s = 59.00^\circ$, $\theta_{sonic} = 55.26^\circ$. 

(d) Analytic results, $\theta_s = 60.00^\circ$, $\theta_{sonic} = 54.27^\circ$. 

(a) Sonic cone. 

(b) Sonic line. 

(Extended text could be added here to describe the figure in detail if needed.)
4.6 Axisymmetric Flow

Axisymmetric flow is characterized by the formation of 3-D, axisymmetric compression and expansion waves. Because axisymmetric flow is symmetrical about the z-axis in a cylindrical coordinate system the flow situation is better described as being ‘quasi-two-dimensional’, just like the conical flow analysis in section 4.5 [5]. Even though there are still only two space coordinates axisymmetric flow is much more difficult to solve because it is essentially a space flow whereas 2-D flows such as the wedge case in section 4.4 are plane flows [6]. The analysis of axisymmetric flow is of great practical importance; almost all high-speed missiles, supersonic aircraft, and projectiles have fuselage or nacelle nosecones that are of an axisymmetric shape [5, 6]. Schlieren images of supersonic flow past axisymmetric bodies in practical applications is given in figures 43(a), 43(b).

![Image](a) Ogive-cylinder at $M = 1.7$ [30].

![Image](b) Hypersonic flow past power-law body at $M_\infty = 8.8$ [30].

**Figure 43:** Shadowgraph images of high-speed flow past bodies of revolution.

There are a number of important axisymmetric bodies applicable to the study of supersonic flow with the three-dimensional cone of section 4.5 belonging to this more general class of three-dimensional bodies. The current research has selected four additional axisymmetric bodies besides the cone for investigation, the tangent ogive, the Sears-Haack body, the paraboloid, and a modified Sears-Haack body shown in figure 44. To describe the supersonic flow over an axisymmetrical body there are three general parameters; the Mach number $M$, the fineness ratio $l/d$, and the hypersonic similarity parameter which is the ratio of the freestream Mach number to the body fineness ratio.

4.6.1 Approximate Analytical Methods

A summary was performed by D. Ehret assessing the accuracy and range of approximate methods to predict the pressure distributions on pointed nonlifting bodies of revolution at zero angle of attack in supersonic flow [7]. The methods reviewed were linearized theory (slender-body theory), second-order theory, the tangent-cone method, conical-shock-expansion theory, and Newtonian theory. The results were
compared to the method of characteristics which can be carried to any degree of accuracy but is generally considered too time consuming to be practical [7].

**Linearized Theory**  A well known linearized or small perturbation theory developed by von Karman and Moore [7] that was already discussed in the presentation of flowfield solution methods in section 1.2. The linearized theory has an advantage over numerical techniques such as the method of characteristics because it is an analytical method, so formulas showing the effects of different parameters in the problem are readily obtained and analyzable [6]. The method assumes frictionless, steady flow, and isentropic conditions along each streamline [6]. An example of the validity of linearized theory is given by a comparison to experiment shown in figure 94 in appendix C.5.

**Second-Order Theory**  Developed by Van Dyke to refine the linearized theory by iteration. The second-order theory is markedly superior to the linearized theory and for values of $M \tan \delta < 0.5$ is almost in perfect agreement with the exact solution [6]. The ratio of the tangent of the body semi-vertex angle to the tangent of the Mach angle must be less than one [7]. An example of the superiority of the second-order theory over the linearized theory is given for the predicted supersonic pressure distributions over a cone in figure 95 in appendix C.5.

**Tangent-Cone Method**  Approximates the flow solutions for cones with identical surface slopes at the points in question. This method is most applicable to the solution of the vertex point since at the tip of a body of revolution the flow is the same as for the initially tangent cone [28].

**Conical-Shock-Expansion Theory**  Eggers and Savin showed that Mach number variation with stream angle downstream of the vertex of an axisymmetrical body approximately reduced o the Prandtl-Meyer equations when the hypersonic similarity parameter K is greater than one. The method first determines the surface pressure of a cone of equal semi-vertex angle as the axisymmetrical body and then flow properties downstream of the vertex are obtained using the Prandtl-Meyer expansion equations [7].
Newtonian Theory  Assumes the component of normal momentum to the surface is lost and the tangential component is unchanged which is achieved in the limit \( M \to \infty \) or \( \gamma \to 1 \). This yields a surface pressure coefficient \( C_p \) which only depends on the local slope of the body. Newtonian theory does not predict the variation in \( C_p \) with Mach number but gives the limiting value for very high Mach numbers [7].

\[
C_p = 2 \sin^2 \theta
\]  

(31)

In 1955, Lester Lees proposed a ‘modified Newtonian’ pressure law given below. The stagnation point on an axisymmetric bodies vertex is the maximum pressure on the body and replaces the constant factor in the original Newtonian expression. The modified Newtonian is used for estimating the pressure distributions on blunt bodies at high Mach numbers and is more accurate than the original Newtonian expression [5].

\[
C_p = C_{p_{\text{max}}} \sin^2 \theta
\]  

(32)

The results of Ehret’s review indicated that the linearized theory is appropriate for low values of the hypersonic similarity parameter \( K \) and that the second order theory extends the linear theory to higher values of \( K \). The second order theory is accurate for the ogive body when the ratio of the tangent of the maximum body angle divided by the tangent of the Mach angle is less than 0.9. The tangent cone method was found to not be generally applicable in any cases. The conical-shock-expansion theory gives engineering accuracy for \( K > 1.2 \). Newtonian theory was found to give relatively accurate results for all non-conical bodies and for cones at high \( K \) [7]. These results can be seen in the comparisons between the calculated pressure distributions over a tangent ogive and modified Sears-Haack body given in figures 92, 93 in appendix C.5. The review also calculated the drag coefficients for the bodies using the approximate methods and compared them to the results from the method of characteristics shown in figure 45; this gives a good graphical representation of the ranges of applicability of each of the approximate methods and is in agreement with the results of [6]. It was found that the range of body shapes, fineness ratios, and Mach numbers for which any one of the approximate theories was applicable was limited. However it was found that for any hypersonic similarity parameter at least one of the methods gave reasonable results. It is therefore important to select the appropriate method given the body geometry and hypersonic similarity parameter for accurate results.

4.6.2  Comparison to Euler 3-D Results

To ascertain the validity of the Euler 3-D results the pressure distributions along a streamwise cross-sectional cut\(^{38}\) of the tangent ogive, Sears-Haack body, paraboloid, and modified Sears-Haack bodies were compared to the results from the method of characteristics presented in [7, 5]. The grids for the bodies were constructed by rotating a 2-D streamwise cross-sectional cut by \( \pi \) radians about the axis of symmetry\(^{39}\). Only the fore half of the body was investigated as in Ehret’s review [7] and [5]. The basic grid geometry is composed of 6 blocks and is shown in figure 91 in appendix C.5. The equations

\(^{38}\)any cut is applicable since the bodies are axisymmetrical

\(^{39}\)it was not required to rotate a full \( 2\pi \) because the flow is symmetrical which saved computational time
Figure 45: Accuracy of approximate methods for integrated pressure drag on axisymmetric bodies from [7].

representing the geometry of each of the axisymmetrical bodies are given with $l$ the body length and $R_{\text{max}}$ the maximum radius or $d/2$.

Effects of Grid Size  Before a full investigation of supersonic axisymmetric flow was undertaken a preliminary analysis on the effects of the grid size was performed. A sample axisymmetric flow case with $M_\infty = 3$ and the modified Sears-Haack body with $l/d = 1.5$ was run with block grids of $8 \times 8 \times 8$, and $16 \times 16 \times 16$\footnote{the dimensions along the $i-j-k$ or $x-y-z$ directions respectively}. The results are shown in figure 46 and reveal that increasing the grid size dramatically sharpens the compression waves as well as resulting in a much smoother body geometry. Most notably it is interesting that although the Mach distribution remains somewhat the same the maximum pressure is much higher and moves forward in the refined grid, this result will become important in the discussion of section 4.6.4. For identical reasons as presented in the analysis of the wedge flow and cone flow the grid size $16 \times 16 \times 16$ was selected as an adequately sized mesh with an acceptable balance between accuracy, computational time, and data manipulation time. A full list of the investigated flow cases is given in table 6 and generally correspond to those investigated by Ehret in the review [7].

Tangent Ogive  A profile shape that is formed by a segment of a circle and described by the following equations for a given maximum radius $R_{\text{max}}$ at the aft end of the body.
Figure 46: Effects of grid size on axisymmetric flow solutions for modified Sears-Haack body at $M = 3.0$ and $l/d = 1.5$. 
Table 6: Investigated axisymmetric flows with grid size $16 \times 16 \times 16$.

<table>
<thead>
<tr>
<th>body type</th>
<th>$M$</th>
<th>$l/d$</th>
<th>$K$</th>
</tr>
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<tr>
<td>Tangent Ogive</td>
<td>2.0</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>2.0</td>
<td>1.0</td>
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<tr>
<td></td>
<td>3.0</td>
<td>3.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Sears-Haack</td>
<td>3.0</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>3.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Mod Sears-Haack</td>
<td>3.0</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
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<tr>
<td></td>
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<td>2.0</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>6.0</td>
<td>4.0</td>
<td>3.0/2.0</td>
</tr>
<tr>
<td>Paraboloid</td>
<td>3.0</td>
<td>1.5</td>
<td>2.0</td>
</tr>
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<td></td>
<td>3.0</td>
<td>3.0</td>
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<tr>
<td></td>
<td>4.0</td>
<td>3.0/2.0</td>
<td>8.0/3.0</td>
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<tr>
<td></td>
<td>4.0</td>
<td>4.0/3.0</td>
<td>3.0</td>
</tr>
</tbody>
</table>

\[
y_{\text{circ}} = \frac{R_{\text{max}}^2 + l^2}{2R_{\text{max}}} \quad (33)
\]

\[
y = \sqrt{y_{\text{circ}}^2 - (l - x)^2 + R_{\text{max}}^2} - y_{\text{circ}} \quad (34)
\]

The pressure distribution along the body surface was found by importing the Euler 3-D data into Matlab\textsuperscript{®}. Only the data in the block containing the fore body shape was analyzed. The pressure distributions within the $16 \times 16 \times 1$ grids were mapped to a square domain with the use of cell indices $(i, j)$ that are then mapped to distances $(x, y)$ which simplifies the data visualization. The pressure distribution is shown for a sample case in figure 48(a) with the change in pressure across the compression waves clearly emphasized.

The coefficient of pressure was then found as shown in figure 48(b) and the $C_p$ distribution along the body surface was plotted and compared to the results from [7] as shown in figures 48(c), 48(d).

**Sears-Haack** Sears-Haack bodies are bodies of minimum wave drag for a given volume, with the cross section being described by the following equation.

\[
y = R_{\text{max}} \left( 1 - \left( \frac{x - l}{l} \right)^2 \right)^{\frac{3}{2}} \quad (35)
\]

It was found through the current analysis that the pressure results for the modified Sears-Haack bodies were identical to the original Sears-Haack bodies. Since only comparison data from [7] for the modified Sears-Haack body were available the discussion of the comparison continues in that section.
Figure 47: Pressure distributions for tangent ogive flow cases.
Figure 48: Method and pressure results for tangent ogive analysis.
Modified Sears-Haack  The modified Sears-Haack body is simply the original Sears-Haack body with the nose vertex replaced by a cone tangent at $x/l = 0.05$. The modified Sears-Haack body was included in the Ehret’s review because the blunt nose of the Sears-Haack body required replacement in order to properly apply the approximate theories. The modified Sears-Haack body is included in the current research for comparison and is described by the same equation as the original Sears-Haack body but with the additional equation describing the fore body vertex.

$$x/l \leq 0.05 \Rightarrow y = Bx$$

(36)

Some sample pressure distributions were already given in figure 46 for the modified Sears-Haack body. The $C_p$ distribution was calculated in an identical manner as with the tangent ogive with the results given in figure 49.

![Figure 49: $C_p$ distribution for modified Sears-Haack body with $M = 3.0, l/d = 3.0$ compared to [7].](image)

Paraboloid  An axisymmetric body that is not blunt but is generated by rotating a segment of a parabola about the Latus rectum and is described by the equation below.

$$y = R_{max} \sqrt{x}$$

(37)

An example solution for the distribution of flow variables throughout the domain is given in figure 50. Unlike the previous axisymmetric bodies, data from Ehret’s review [7] is not available for the paraboloid, instead data relating the pressure ratio $P/P_0$ along the bodies surface was given by [5] and compared to the current results in figure 51. The stagnation pressure $P_0$ was taken as the maximum pressure in the Euler 3-D simulation despite the fact that the actual value should be at the bodies vertex, this discrepancy is related to the pressure solution due to grid size and will be discussed in section 4.6.4.
Figure 50: Flow variable variation for paraboloid with $M = 3.0$, $l/d = 1.5$. 

(a) Pressure distribution. 

(b) Mach number distribution. 

(c) Nose pressure distribution.
4.6.3 Comparison between Axisymmetric Bodies

To identify the differences in the pressure distributions between the different axisymmetric bodies a comparison was performed for the same operating conditions and body sizes, \( M = 3.0 \) and \( l/d = 3.0 \), which gives a hypersonic similarity parameter of \( K = 1.0 \), the results are given in figure 52.

The comparison shows that the paraboloid has the greatest \( C_p \) of all the bodies. The paraboloid also has the quickest decrease in \( C_p \) along its length but never has a negative \( C_p \). The modified Sears-Haack body and the original Sears-Haack body have identical \( C_p \) distributions and have a larger \( C_p \) than the tangent ogive. Near the end of the Sears-Haack bodies \( C_p \) becomes the most negative of all the bodies. The tangent ogive has the smallest rise in \( C_p \) and the smallest maximum \( C_p \), and also has the most gentle decrease in \( C_p \) after the maximum is reached. From these results a preliminary analysis would indicate that the paraboloid has the highest drag and the tangent ogive has the lowest drag. However it is important to note that the Sears-Haack bodies were designed as full bodies of revolution [6] and not to include only the fore half as is the case in this analysis, therefore they may overall have lower drag than the tangent ogive despite the current results.

4.6.4 Validity of Current Results

It was found in section 4.6.2 that when the grid size increased the maximum pressure on the bodies surface also increased as well as moved closer to the fore body vertex. The \( C_p \) distribution comparisons for each of the axisymmetric bodies have shown that the simulated pressure near the body vertex from the Euler 3-D results is significantly lower than expected. These pressure differences are due to the grid size near the fore vertex of the bodies being too coarse. Due to the coarse grid size the compression waves are not well defined in this region and the pressure does not significantly increase until some distance downstream from the vertex. However once the compression wave effects are accounted for it was found that the \( C_p \) distributions were relatively accurate for the remainder of the bodies surface for all axisymmetric cases.
To investigate this phenomenon two cases were run with an increased grid size of $32 \times 32 \times 32$. As the results show in figure 53 the pressure does increase and move towards the fore body vertex with the increase in grid size and the $C_p$ distributions become more accurate both near the fore body vertex and also aft along the body surfaces. The results indicate that in the limit of increasing the grid size the simulated Euler 3-D solution will converge to the solution of the method of characteristics from [7, 5], in particular a large number of cells are required near the tip of axisymmetric bodies to capture the sudden, drastic increase in pressure.

**Figure 52:** Comparison between $C_p$ distributions for axisymmetric bodies, $M = 3.0, l/d = 3.0$. 
(a) Pressure distribution for modified Sears-Haack body at \( M = 3.0, l/d = 3.0 \).

(b) Fore body vertex pressure distribution for modified Sears-Haack body at \( M = 3.0, l/d = 3.0 \).

(c) \( C_p \) distribution for modified Sears-Haack body at \( M = 3.0, l/d = 3.0 \) with comparison to [7].

(d) \( C_p \) distribution for tangent ogive body at \( M = 2.0, l/d = 2.0 \) with comparison to [7].

**Figure 53:** Effects on pressure and \( C_p \) distributions for further refinement of grid.
5 Simulation of Supersonic Flow past SST Aircraft

The simulation was conducted by first modeling the flat-top wing/body aircraft configuration of Eggers as presented in section 3.1 and shown in figure 54. The aircraft body was constructed using a half-cone with the desired semi-vertex angle $\theta_c$ and the wings were swept back by the desired sweep angle $\theta_{\text{sweep}}$. The trailing edge of the wings connects the wingtips with the aft-most portion of the aircraft’s body as in the aircraft design of Eggers [3]. The aircraft’s wings do not converge to a point at the wingtips as to prevent solution errors, but have a finite width defined by the wingtip length $l_{\text{wingtip}}$. To simulate an angle of attack the incoming flow is deflected by the desired angle $\alpha$.

![Diagram of aircraft configuration](image1)

(a) Complete flat-top, half-body aircraft configuration with drooping wingtips from [3].

![Simulation grid](image2)

(b) Complete simulation grid consisting of 8 blocks.

![Grid and geometry](image3)

(c) Cut-away view of the simulation grid showing aircraft geometry.

(d) Top profile of aircraft geometry showing swept back wings and conical body.

**Figure 54:** Geometric aircraft representation in simulation grid with comparison to original design, green blocks represent atmosphere and red blocks represent aircraft geometry, grid size $8 \times 8 \times 8$. 
5.1 Euler 3-D Results

There was insufficient time to thoroughly investigate and properly simulate the supersonic flow past the aircraft geometry shown in figure 54 for a wide range of cone semi-vertex angles $\theta_c$, wing sweeps $\theta_{\text{sweep}}$, wingtip lengths $l_{\text{wingtip}}$, Mach numbers $M$ and angles of attack $\alpha$. However a preliminary investigation was completed at a grid size of $16 \times 16 \times 16$ for two flow cases for the range $\alpha = 0.0^\circ, 2.0^\circ, 5.0^\circ, 10.0^\circ$.

The analysis was begun with the flow case $M = 2.0, \theta_c = 5.0^\circ$. The sweep on the wings was chosen to coincide with the predicted shock wave from the conical flow results of section 4.5.1 and was found to be $\theta_{\text{sweep}} = 30^\circ$. The wingtip length was chosen to be $l_{\text{wingtip}} = 0.1$ as an over-estimate to avoid the possibility of flow solution errors due to the grid distribution. The flow was run for the range $\alpha = 0.0^\circ, 2.0^\circ, 5.0^\circ, 10.0^\circ$ with the results shown in figures 55, 56, 58.

To investigate the effects of the Mach number and the body semi-vertex angle both these variables were increased for the following analysis with $M = 3.0, \theta_c = 10^\circ$. The wing sweep was kept aligned with the predicted shock angle at $\theta_{\text{sweep}} = 22^\circ$ and the wingtip length was kept at $l_{\text{wingtip}} = 0.1$. The flow was run for the range $\alpha = 0.0^\circ, 2.0^\circ, 5.0^\circ, 10.0^\circ$ with the results shown in figures 57, 59.

![Figure 55](image.png)

(a) Pressure distribution. (b) Mach distribution.

**Figure 55:** Aircraft simulation results for $M = 2.0, \theta_c = 5.0^\circ, \alpha = 0.0^\circ$, aft-view.

5.2 Interpretation of Results

The results from the Euler 3-D simulation show a number of interesting flow phenomena. From the flow solutions at $\alpha = 0.0^\circ$ the conical body of the aircraft produces a similar solution to the cone analysis presented earlier in section 4.5.3 but with the wings now acting to 'trap' the compression below the aircraft. However as soon as a non-zero $\alpha$ is achieved the flow situation changes completely. In the case for
Figure 56: Downstream Mach distribution aft-view for $M = 2.0, \theta_c = 5.0^\circ$.

Figure 57: Downstream Mach distribution aft-view for $M = 3.0, \theta_c = 10.0^\circ$. 
Figure 58: Pressure distribution slices with axial distance, $M = 2.0, \theta_c = 5.0^\circ$ for various $\alpha$. 

(a) $x/l = 0.3, \alpha = 0.0^\circ$.  
(b) $x/l = 0.6, \alpha = 0.0^\circ$.  
(c) $x/l = 1.0, \alpha = 0.0^\circ$.  

(d) $x/l = 0.3, \alpha = 2.0^\circ$.  
(e) $x/l = 0.6, \alpha = 2.0^\circ$.  
(f) $x/l = 1.0, \alpha = 2.0^\circ$.  

(g) $x/l = 0.3, \alpha = 5.0^\circ$.  
(h) $x/l = 0.6, \alpha = 5.0^\circ$.  
(i) $x/l = 1.0, \alpha = 5.0^\circ$.  

(j) $x/l = 0.3, \alpha = 10.0^\circ$.  
(k) $x/l = 0.6, \alpha = 10.0^\circ$.  
(l) $x/l = 1.0, \alpha = 10.0^\circ$.  

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Figure 59: Pressure distribution slices with axial distance, $M = 3.0, \theta_c = 10.0^\circ$ for various $\alpha$. 

University of Toronto
the relatively small body (semi-vertex angle $\theta_c = 5.0^\circ$) expansion waves decrease the pressure on the upper surface of the aircraft and compression waves increase the pressure below the aircraft as shown in figure 58. It is interesting to note that these waves become stronger downstream along the body and are most prominent near the wingtips of the aircraft. As the angle of attack is further increased the region of expansion above the aircraft continues to increase and becomes more elongated while the region of compression flattens out in the middle. These flow phenomena are also clearly shown in the downstream Mach distributions shown in figure 56. In this flow case the body appears to have a relatively small impact on the flow at any $\alpha > 0.0$. When the size of the body is increased as is the case for the second flow scenario (semi-vertex angle $\theta_c = 10.0^\circ$) the body begins to have a very significant impact for all $\alpha$ as shown in figures 57, 59. The influence of the body causes an increased compression effect below the aircraft wings and is most prominent along the center of the lower aircraft surface.

It was interesting to find that the flow expansion on the upper surface of the aircraft for $\alpha > 0.0$ was most prominent near the wingtips and was much weaker along the centerline of the body. This is possibly a three dimensional flow phenomenon such as the vortex formation along the wing leading edges of delta-wing aircraft at high $\alpha$ [4].

5.3 Future Aircraft Simulation Work

The flow situation for the current aircraft geometry is fairly complex for $\alpha > 0.0$ and warrants a more detailed investigation with a much finer grid size. It is proposed for future research to investigate the current aircraft geometry using the HPACF or Scinet computing facilities which would allow a much more detailed investigation and analysis. Furthermore the ultimate purpose of the current research was to determine the L/D ratios for aircraft utilizing compression lift which requires the integration of pressure forces over the entire aircraft surface. This goal was unobtainable for the current research due to time restrictions but the simulation geometry is fully prepared and it would be relatively simple to continue the current research and obtain the desired L/D ratios.

It would also be interesting to extend the current simulation geometry to incorporate drooping wingtips like those shown in the aircraft configuration of Eggers in figure 54(a) and as employed on the XB-70 aircraft as shown in figure 1.
6 Conclusions

The design of a successful SST aircraft has been shown to be an extremely difficult technical problem as well as an important goal. The current thesis has focused on the aerodynamic challenge of increasing the supersonic efficiency, measured by the L/D ratio, of SST aircraft. The current research has focused on the concept of compression lift developed by NACA in the 1950’s which has never thoroughly been applied to the design of a SST aircraft.

To place the primary analysis tool, the UTIAS Euler 3-D simulation code, into context a review of the relevant supersonic analysis methods and tools was performed. The Euler 3-D code belongs to the CFD category which currently has been replacing the older methods due to its accuracy and range of applicability. A review of the most significant SST aircraft and SST aircraft designs was performed to gain insight into the aerodynamic design of these aircraft. It was found that most recent designs generally opt for a sleek arrow-wing design with separate engine nacelles rather than the integrated compression lift design of the older XB-70 aircraft. A review of the concept of compression lift involving a full derivation of the original research was performed. In addition a number of other applicable aerodynamic theories were reviewed to gain insight into the design of supersonic compression lift aircraft.

To validate the UTIAS Euler 3-D code a range of simple supersonic geometries were simulated and compared to the analytic results. The simulation of 1-D plate flow had significant error because of convergence issues, possibly due to the flow being unsteady and rotational. The simulation of 2-D wedge flow compared shock wave angles and the shock polar and was relatively accurate when compared to the analytical results. The wedge analysis also included the use of AMR and the effects of further grid refinement which showed that the solution was convergent for increasing grid size. Before simulating the 3-D conical flow the analytical equations for the Taylor-Maccoll solution were derived and implemented to solve for the shock wave angles and the conditions for isentropic compression with post-shock sonic lines. The simulated conical flow results were found to be relatively accurate in comparison to the analytical results and the analysis included the use of AMR and grid refinement which also showed the solution was convergent for increasing grid size. A number of axisymmetric bodies were then investigated and the $C_p$ distributions were calculated and compared to the results of the method of characteristics. The solutions had significant error but a grid refinement analysis showed that again the solution was convergent for increasing mesh size. In conclusion the Euler 3-D code was found to quantitatively simulate all the simple supersonic flow cases and to provide adequate quantitative results given the grid mesh sizes. The fact that the solutions were found to be convergent with increasing mesh size further qualified the simulated solutions.

A preliminary investigation of the supersonic flow past a SST aircraft utilizing compression lift was performed. The aircraft geometry was modeled to conform to the design presented by Eggers from the original NACA research on compression lift. The preliminary results indicate that a small aircraft body has little effect on the flow but that a larger aircraft body significantly contributes to the compression underneath the aircraft for all $\alpha$. It was proposed that further analysis using the HPACF or Šćinet is required for more detailed and accurate results. Furthermore the final goal of computing the L/D ratios for compression lift aircraft requires further work but since the simulation geometry is fully prepared it would be a relatively simple task to continue the current research and obtain the desired results.
References


A Appendix: Supersonic Transport (SST) Aircraft

A.1 SST Research, Challenges, and Tools

**Figure 60:** Chronology of SST research [13].

**Figure 61:** Application of wire-mesh program to SCAT-15F configuration [1].
A.2 SST Aircraft Configurations

(a) Lockheed L-2000 Model [1].

(b) Lockheed L-2000 three-view [16].

Figure 62: Lockheed L-2000 Supersonic Civilian Transport.

(a) Tu-144 in flight [16].

(b) Tu-144 three-view [16].

Figure 63: Tupolev Tu-144 Supersonic Civilian Transport.
Figure 64: Concorde Supersonic Civilian Transport.

Figure 65: High Speed Civil Transport (HSCT) reference H [39].

Figure 66: High Speed Civil Transport (HSCT) predecessor by McDonnell Douglas [1].
B Appendix: Supersonic Theory and Analysis

B.1 Compression Lift

Figure 67: Effect of cone semi-vertex angle on L/D ratios for conical flat-top aircraft configurations at Mach 5.0 from [3].
**Figure 68:** Experimental lift and L/D ratios for flat-top aircraft model with wingtip deflections from [3].

**Figure 69:** Experimental lift and L/D ratios for flat-top, flat-bottom, and symmetrical aircraft models from [19].
Figure 70: Effects of drooping wingtips on flat-top aircraft configurations from [19].

Figure 71: Body wave drag increment due to wing anhedral from [22], only poor reproduction quality available.
B.2 Arrow-Wings

**Figure 72:** Effect of body diameter on maximum L/D ratio of arrow-wing aircraft from [23].
(a) Effect of body volume on L/D ratio at $M \cdot 2.94$.

(b) Effect of total volume on L/D ratio at $M \cdot 2.94$.

**Figure 73:** Comparison of the effects of body and wing volume on L/D ratios of arrow-wing aircraft from [23].
C Appendix: Shock Waves

C.1 Introduction to Shock Wave Theory

Figure 74: Pressure vs. specific volume for isentropic and Hugoniot curves [5].
C.2 Plate Flow

Figure 75: Un-converged supersonic flow at $M_\infty = 3.0$ past a 2-D wedge.
C.3 Wedge Flow

(a) Flowfield situation for oblique shock polar [5].

(b) Shock polar geometry [33].

Figure 76: Shock polar flowfield and geometry.

Figure 77: Meyer’s original oblique shock polar relating $P_1/P_0$ to $P_2/P_0$ with $\omega$ replacing $\theta$ [32].
Figure 78: Basic grid outline and block structure for wedge flow.
Figure 79: Refined wedge flow simulations for grid size $32 \times 32 \times 4$. 

(a) Mach 4.0, $\theta_c = 15.0^\circ$.

(b) Mach 5.0, $\theta_c = 10.0^\circ$. 

Figure 79: Refined wedge flow simulations for grid size $32 \times 32 \times 4$. 

C.4 Conical Flow

Figure 80: Comparison of calculated surface pressures with wind tunnel observations for conical flow [27].

Table 7: Table of results for Euler 3D isentropic compression with transition in conical flow.

<table>
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<th>$\theta_{sonic}$</th>
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<td>38.00°</td>
<td>55.26°</td>
</tr>
<tr>
<td>3</td>
<td>59.55°</td>
<td>46.00°</td>
<td>58.70°</td>
</tr>
<tr>
<td>3</td>
<td>61.07°</td>
<td>47.00°</td>
<td>60.30°</td>
</tr>
<tr>
<td>4</td>
<td>61.88°</td>
<td>50.00°</td>
<td>64.84°</td>
</tr>
<tr>
<td>5</td>
<td>62.50°</td>
<td>52.00°</td>
<td>69.10°</td>
</tr>
</tbody>
</table>
Figure 81: Comparison of calculated shock wave angles with wind tunnel observations for conical flow [27].

Figure 82: Supersonic flow past a 30 deg semi-vertex bullet at $M = 2.11$ [35].
Figure 83: Supersonic flow past a 30 deg semi-vertex bullet at $M = 1.26$ [35].

<table>
<thead>
<tr>
<th>$U/a$</th>
<th>1.09</th>
<th>1.04</th>
<th>1.07</th>
<th>1.22</th>
<th>1.39</th>
<th>1.81</th>
<th>2.39</th>
<th>3.33</th>
<th>5.46</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>0.69</td>
<td>0.37</td>
<td>0.12</td>
<td>0.13</td>
<td>0.17</td>
<td>0.26</td>
<td>0.38</td>
<td>0.58</td>
<td>0.84</td>
</tr>
</tbody>
</table>

Figure 84: Ratio $F$ of pressure change in shock wave to the Total pressure change from undisturbed air to solid surface [35].
(a) Comparison between critical speeds of cone and wedge [6].

(b) Comparison of shock wave angles for wedges and cones at Mach 2 [5].

**Figure 85:** Comparisons between supersonic cone and wedge flow.

**Figure 86:** Upstream Mach numbers for shock attachment for wedges and cones in supersonic flow [28].
Figure 87: Comparison between Taylor-Maccoll and Karman-Moore solutions for conical flow [35].

Figure 88: Apple curves for flow past cone with M = 1.5 [6].
Figure 89: Basic grid outline and block structure for cone flow.
Figure 90: Refined cone flow simulations for grid size $32 \times 32 \times 32$.

(a) Mach 2.0, $\theta_c = 10.0^\circ$.

(b) Mach 3.0, $\theta_c = 30.0^\circ$. 
C.5 Axisymmetric Flow

Figure 91: Basic grid geometry for axisymmetric flow.
Figure 92: Comparison of calculated pressure distributions from approximate methods on tangent ogive at $K = 1$, $l/d = 3$, $M = 3$ from [7].
Figure 93: Comparison of calculated pressure distributions from approximate methods on modified Sears-Haack body at $K = 1$, $l/d = 3$, $M = 3$ from [7].
Figure 94: Comparison between experiment and linearized theory of pressure distribution on A-4 missile from [6].
Figure 95: Comparison of calculated pressure distributions from approximate methods on cone body from [6].